Section 3.5

Gaussian distribution

(also called normal distribution)

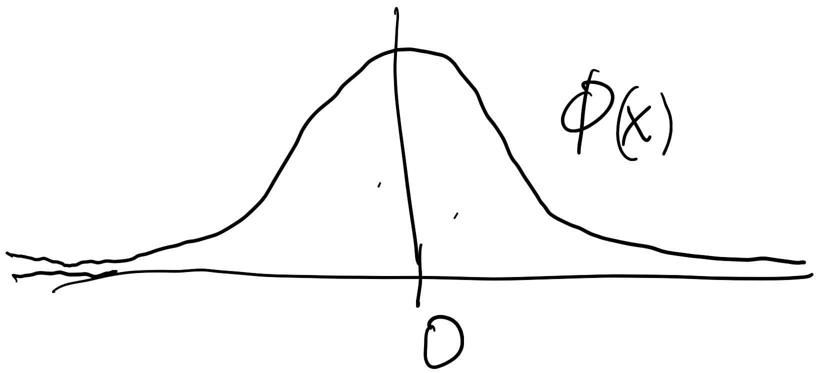
Most simple version of Gaussian dist.

Def: A r.v. Z has the *standard normal dist*. (or standard Gaussian dist.) if it has density function

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$

Notation: $Z \sim N(0,1)$.

Plot of pdf:



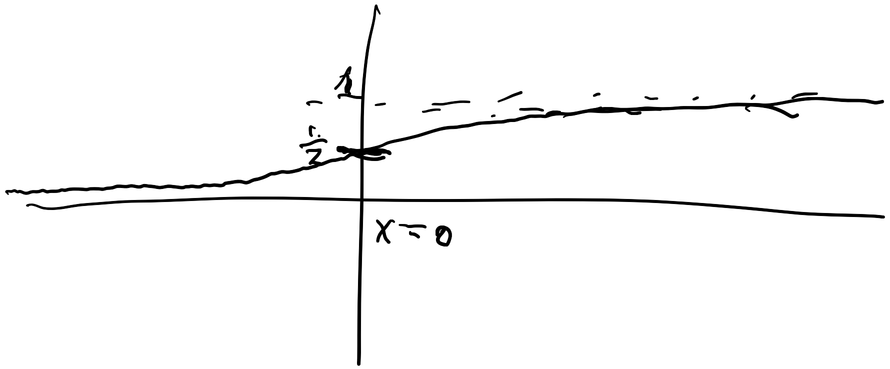
Standard normal

Cdf also gets a special symbol:

$$\Phi(x) = \underline{P(Z \leq x)} = \int_{-\infty}^{x} \phi(s)ds = \boxed{\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds}$$

No closed form soln

Plot of CDF:



Why the factor of $\frac{1}{\sqrt{2\pi}}$?

Fact: $\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$ so $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = 1$

Pf: $\left(\int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \int_{-\infty}^{\infty} e^{-y^2/2}\,dy$

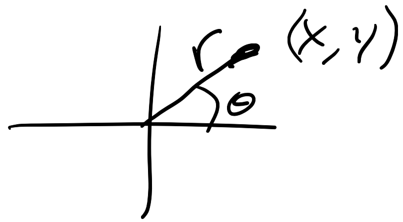
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\cdot e^{-y^2/2}\,dx\,dy = \iint_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\,dx\,dy$$

Change to polar:

$dx\,dy = r\,dr\,d\theta$

$$= \int_0^{2\pi}\int_0^{\infty} e^{-\frac{r^2}{2}}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\cdot d\theta$$



$= \int_0^{2\pi} (0 - \cancel{-1}) \, d\theta = 2\pi$ ✓

Why called "*standard* normal"

- **Fact:** If $Z \sim N(0,1)$ has $EZ = 0$, $Var(Z) = 1$

This is why Z is called standard normal.
To get a whole family of (non-standard) normal r.v.s, just consider $X = \sigma Z + \mu$ for different values of σ and μ.

Let us find the dist of X. First:

$EX = \mathbb{E}[\sigma Z + \mu] = \sigma \cancel{\mathbb{E}[Z]}^{0} + \mu = \mu.$

$Var(X) = Var(\sigma Z + \mu) = \sigma^2 \cancel{Var(Z)}^{1} = \sigma^2$

Cdf and pdf of $X = \sigma Z + \mu$

Cdf: $F(s) = P(X \leq s) = P(\sigma Z + \mu \leq s) = P\left(Z \leq \frac{s-\mu}{\sigma}\right) = \Phi\left(\frac{s-\mu}{\sigma}\right)$

Pdf: $f(s) = \frac{d}{ds} F(s) = \frac{d}{ds} \Phi\left(\frac{s-\mu}{\sigma}\right) = \frac{1}{\sigma} \cdot \phi\left(\frac{s-\mu}{\sigma}\right)$

$= \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{s-\mu}{\sigma}\right)^2}$

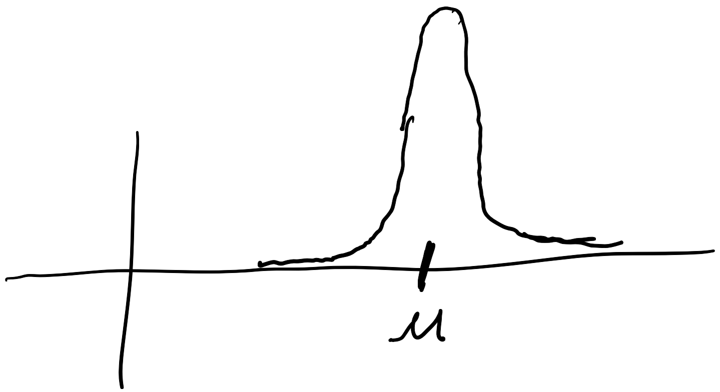
(General) Normal dist (or Gaussian dist)

Def: A r.v. Z has the normal dist. (or Gaussian dist.) if it has density function

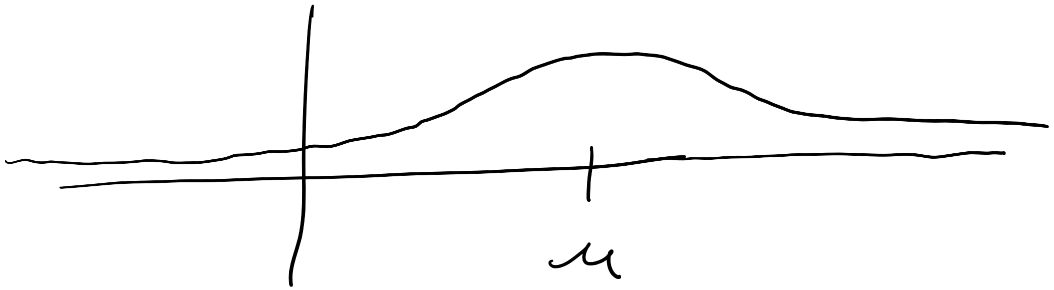
$$\mathrm{f}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad \text{for some } \mu \in \mathbb{R},\ \sigma > 0.$$

Notation: $\mathrm{Z} \sim N(\mu, \sigma^2)$.

Plot: σ small



σ large



Rescaling, translating → still Gaussian

Fact: If $X \sim N(\mu, \sigma^2)$, and $Y = aX + b$, then $Y \sim N(a \cdot \mu + b,\ a^2 \cdot \sigma^2)$.

In particular, $\frac{X-\mu}{\sigma} \sim N(0, 1)$

Ex) $X \sim N(-3, 5)$. What is $P(4(X+5) > 2)$?

$$P(4(X+5) > 2) = P\left(X > \frac{2}{4} - 5\right) = P\left(X > -\frac{9}{2}\right)$$

$$= P\left(N(-3,5) > -\frac{9}{2}\right) = P\left(\left[-3 + \sqrt{5}\,N(0,1)\right] > -\frac{9}{2}\right)$$

$$-3 + \sqrt{5}\,N(0,1) \overset{dist}{=} N(-3,5)$$

$$= P\left(\sqrt{5}\,N(0,1) > \cancel{3 - \frac{9}{2}}\ -3/2\right)$$

$$= P\left(N(0,1) > \frac{-3/2}{\sqrt{5}}\right)$$

$$= 1 - P\left(N(0,1) \le \frac{-3/2}{\sqrt{5}}\right) = \boxed{1 - \Phi\left(\frac{-3/2}{\sqrt{5}}\right)}$$