Section 3.3

Expectation

Ex) Casino game: You roll a die, if it comes out 3, you win \$6 (otherwise, you win nothing). How much would you be willing to pay to play this game?

What if game is more complicated? How do you decide how much is a fair price to play?

Expected value

Def: The expected value (or mean or expectation) of r.v. X is:

Discrete case: $\mathbb{E}X := \sum_k k\, p(k)$

Continuous case: $\mathbb{E}X := \int_{-\infty}^{\infty} x\, f(x) dx$

Meaning: Weighted average of values X can take.

Ex) Roll die, outcome is X. $\mathbb{E}X = \sum_{i=1}^{6} i \frac{1}{6} = \frac{1}{6} \cdot \frac{6(6+1)}{2} = \frac{7}{2} = 3.5$

$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Ex) Let X be the amount of money you gain in previous game, so

$$p(0) = \frac{5}{6}, \qquad p(6) = \frac{1}{6}$$

Then, $EX = \underline{0 \cdot \frac{5}{6} + 6 \cdot \frac{1}{6} = 1}$

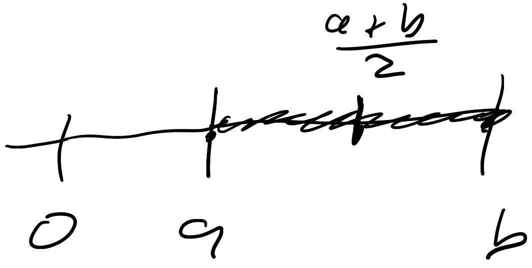
Ex) Flip 3 coins, X = number of heads.

$$X \sim \underline{\text{Bin}}(3, 1/2)$$

$$EX = \underline{\binom{3}{0}} \cdot \frac{1}{2^3} \cdot 0 + \binom{3}{1} \cdot \frac{1}{2^3} \cdot 1 + \binom{3}{2} \cdot \frac{1}{2^3} \cdot 2 + \binom{3}{3} \cdot \frac{1}{2^3} \cdot 3$$

$$= \frac{1 \cdot 0 + 3 \cdot 1 + 3 \cdot 2 + 1 \cdot 3}{8} = \frac{12}{8} = \boxed{\frac{3}{2}}.$$

Ex) $X \sim Unif[a, b]$



$$\mathbb{E}X = \int_a^b \frac{1}{b-a} \cdot x\,dx$$

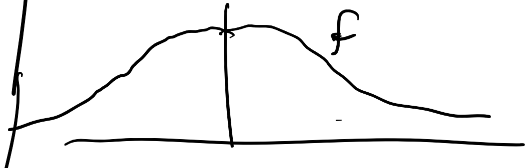
$$= \frac{1}{b-a} \cdot \frac{x^2}{2}\Big|_a^b = \frac{1}{b-a} \cdot \frac{(b^2-a^2)}{2} = \frac{\cancel{(b-a)}(b+a)}{\cancel{b-a} \quad 2} \quad \checkmark$$

Ex) X has pdf $f(x) = \begin{cases} \frac{1}{x^2}, & x \geq 1 \\ 0, & x < 1 \end{cases}$

$$\mathbb{E}X = \int_1^\infty \frac{1}{x^2} \cdot x\,dx = \ln(x)\Big|_0^\infty = \infty.$$

Recall if X has pdf $f(x) = \frac{1}{\pi(1+x^2)}$

$\mathbb{E}X$ DNE



$\int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)}\,dx$ DNE

Ex) X has pdf $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$

$$\mathbb{E}X = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x\, e^{-x^2/2}\,dx = \frac{1}{\sqrt{2\pi}} \left(-e^{-\frac{x^2}{2}}\right)\Big|_{-\infty}^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}(0 - 0) = 0$$

Expected value of a sum

Fact: $\mathbb{E}[X_1 + X_2 + \ldots + X_n] = \mathbb{E}X_1 + \mathbb{E}X_2 + \ldots + \mathbb{E}X_n$
(even if r.v.s $X_1, X_2, \ldots, X_n$ are dependent!)

Ex) $X \sim Bin(n, p)$. $\mathbb{E}X = ?$

Recall $X_1 + X_2 + \ldots + X_n \sim Bin(n, p)$

indep. Bern(p)

$$\mathbb{E}X = \mathbb{E}[X_1 + \ldots + X_n] = \mathbb{E}X_1 + \mathbb{E}X_2 + \ldots + \mathbb{E}X_n$$
$$= p + p + \ldots + p = \boxed{np}$$

X = number of pairs of students who share a birthday, out of n students.

$$X = \sum_{\substack{i \neq j \\ i,j}} \mathbb{1}_{A_{i,j}}$$

$A_{i,j}$ = { students i & j share b-day }

$$\mathbb{1}_{A_{i,j}} = \begin{cases} 1 & \text{if } A_{i,j} \text{ occurs} \\ 0 & \text{else} \end{cases}$$

$$\mathbb{E} X = \sum \mathbb{E}\left[\mathbb{1}_{A_{i,j}}\right]$$

$$= \sum_{\text{pairs}} P(A_{ij}) = \sum_{\text{pairs}} \frac{1}{365} = \boxed{\frac{\binom{n}{2}}{365}}$$

Expected value of $X \sim Geom(p)$

$\mathbb{E}X = \frac{1}{p}$ see last lecture.

Expectation of non-negative r.v.

Fact: If X only takes non-negative values, then

- Discrete case: $\mathbb{E}X := \sum_{k=1}^{\infty} P(X \geq k)$, assuming X takes values in $\{0, 1, 2, \ldots\}$.
- Continuous case: $\mathbb{E}X := \int_0^{\infty} P(X \geq x)\, dx$

Expected value of $X \sim Geom(p)$

Expected value of a function of a r.v.

Fact: Let the range of r.v. X be contained in the domain of real-valued function g. Then,

- Discrete case: $\mathbb{E}g(X) := \sum_k g(k)\, p(k)$
- Continuous case: $\mathbb{E}g(X) := \int_{-\infty}^{\infty} g(x)\, f(x)dx$

Moments of a r.v.

Def: The n^{th} *moment* of r.v. X is $\mathbb{E}X^n$.

Ex) Let $X \sim Unif[0,1]$. What is n^{th} moment of X?

$$\mathbb{E}X^n = \int_0^1 \underbrace{x^n}_{g(x)} \cdot \underbrace{1}_{f(x)} \, dx = \frac{x^{n+1}}{n+1}\Big|_0^1 = \boxed{\frac{1}{n+1}}$$