Chapter 3

Random variables

Announcements:

1. HW due Fri 10 pm
2. Guest lecturer Wed & next Wed
3. OH start this week
4. Practice quiz on canvas

Section 3.1

Probability distributions

Abbreviation: distribution = dist.

Recall:

Probability dist. for r.v. X is set of all probabilities $\{P(X \in B)\}$

Probability mass function (pmf): For discrete r.v. $\mathrm{p}(\mathrm{k}) = \mathrm{P}(\mathrm{X} = \mathrm{k})$

Relationship: $P(X \in B) = \sum_{k \in B} p(k)$

Ex) Flip 2 coins. $X =$ number of heads

Pmf: $p(0) =$ $\underline{1/4}$, $p(1) =$ $\underline{1/2}$, $p(2) =$ $\underline{1/4}$ $\qquad X \sim Bin(2, 1/2)$

Probability distribution:

$$P(X \in \{0,1\}) = \underline{p(0) + p(1)} = \underline{\frac{1}{4} + \frac{1}{2} = \frac{3}{4}}$$

$$P(X \in \{0,2\}) = \underline{p(0) + p(2)} = \underline{\frac{1}{2}}$$

$$P(X \in \{0,1,2\}) = \underline{p(0) + p(1) + p(2)} = \underline{1}$$

$$\vdots$$

$P(X \in B)$ $\qquad P(X \leq 1.5) = P(X \in \{0,1\}) = \frac{3}{4}$

Q) What if X is not discrete?

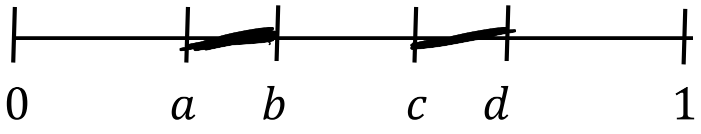
Ex) X is uniformly chosen from $[0,1]$.

$$P(X = k) = \underline{\ \ 0\ \ }$$

How to find $P(X \in B)$?

Ex) $P(X \in [.5, 1]) = \underline{\ \ 1/2\ \ }$

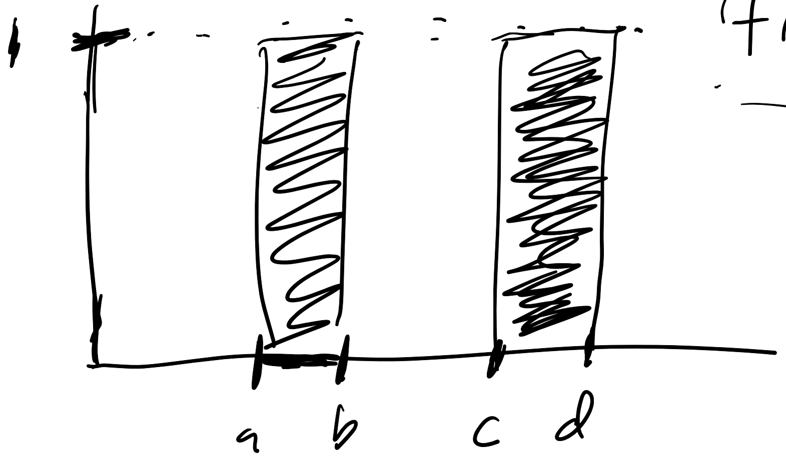
Ex) X is uniformly chosen from $[0,1]$



$B = [a,b] \cup [c,d]$

$$P(X \in B) = P(X \in [a,b]) + P(X \in [c,d])$$

$= b - a + d - c$



$f(x) = 1$

Area under curve

$= P(X \in B)$

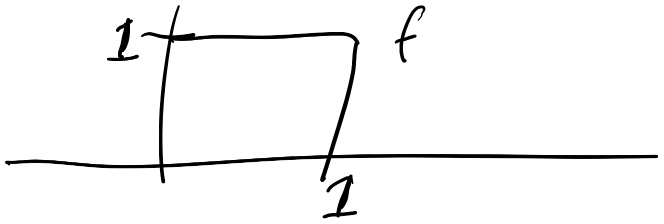
Probability density function

Def: We say that a r.v. X has *probability density function* (pdf) f if

$$P(X \leq a) = \int_{-\infty}^{a} f(x)\, dx$$

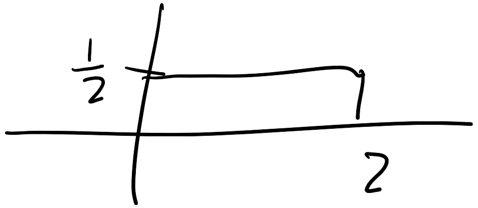
Ex) If X is drawn uniformly from $[0,1]$ then X has pdf

$$f(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & x \notin [0,1] \end{cases}$$



Q) What if X is drawn uniformly from $[0, 2]$?

$$f(x) = \begin{cases} 1/2 & x \in [0, 2] \\ 0 & x \notin [0, 2] \end{cases}$$



Note: Not every random variable has a pdf.

E.g., discrete random variables don't have pdfs

If X has pdf f then:

- We call X a *continuous r.v.*
- $P(X \in [a,b]) = \int_a^b f(x)dx = P(X \in (a,b))$
- $P(X \in B) = \int_B f(x)dx$
- $P(X = k) = \int_k^k f(x)dx = 0$ i.e., $P(X$ takes one particular value) = 0
- Pdf must satisfy $\int_{-\infty}^{\infty} f(x)dx = 1$, $f(x) \geq 0$

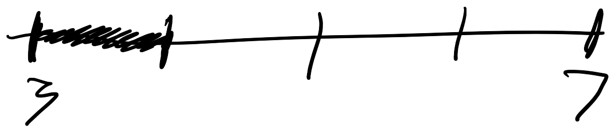
Ex) Uniform r.v.

Def: Let X have pdf

$$f(x) = \begin{cases} \frac{1}{b-a} & x \in [a,b] \\ 0 & x \notin [a,b] \end{cases}$$

Then we say X has uniform distribution on $[a, b]$, i.e., $X \sim Unif[a, b]$

Ex) Let $X \sim Unif[3,7]$. What is $P(X \in [3,4])$?



$$f(x) = \begin{cases} \frac{1}{7-3} = \frac{1}{4} & x \in [3,7] \\ 0 & x \notin [3,7] \end{cases}$$

$$P(X \in [3,4]) = \int_3^4 f(x)\,dx = \int_3^4 \frac{1}{4}\,dx = \frac{1}{4}$$

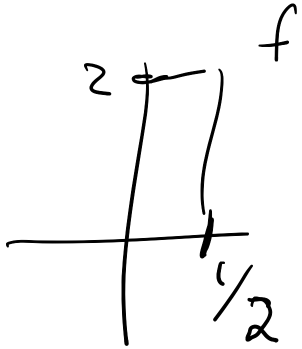
$$P(X \in [6,8]) = \int_6^8 f(x)\,dx = \int_6^7 \frac{1}{4}\,dx = \frac{1}{4}$$

since $f(x) = 0$ $x > 7$

Note: $f(x)$ can be > 1

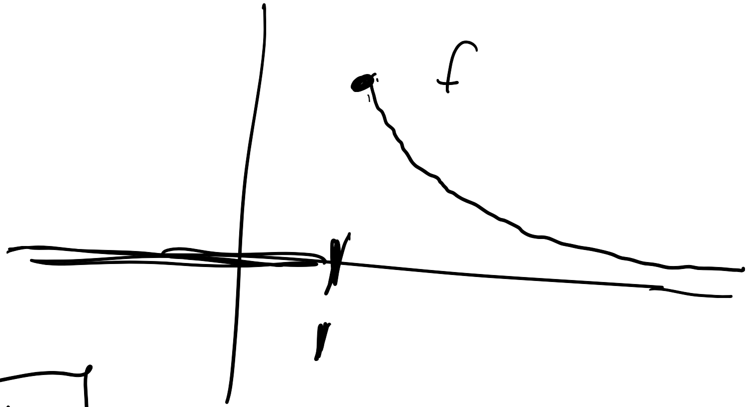
Ex) $X \sim Unif\left[0, \frac{1}{2}\right]$

$$f(x) = \begin{cases} 2 & x \in \left[0, \frac{1}{2}\right] \\ 0 & x \notin \left[0, \frac{1}{2}\right] \end{cases}$$



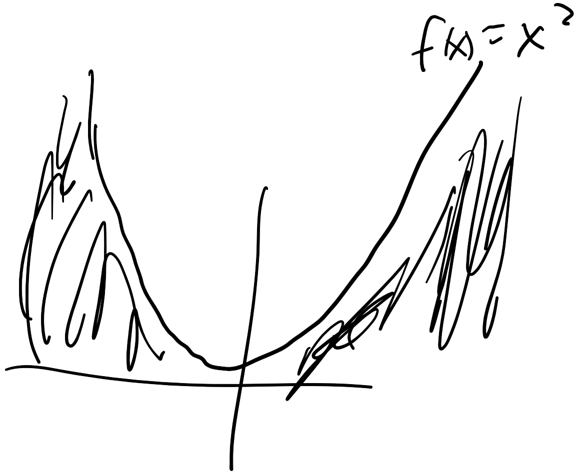
Non-uniform example:

Q) Let X have pdf $f(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & x < 1 \end{cases}$



$P(X > 5) =$ $\int_5^\infty \frac{1}{x^2}\,dx = -\frac{1}{x}\Big|_5^\infty = \boxed{\frac{1}{5}}$

Careful: Is it really a pdf?



Q) Let X have pdf $f(x) = x^2$ for all $x \in \mathbb{R}$

$$\cancel{P(X < 4) = \underline{\hspace{4cm}}}$$

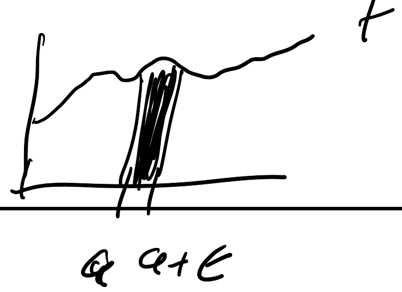
f not a pdf.

$$\int_{-\infty}^{\infty} x^2 \, dx = \infty$$

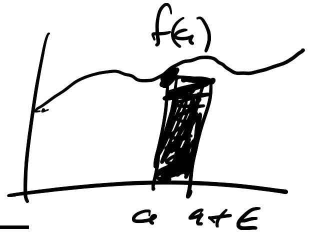
Intuitive meaning of $f(x)$

$f(a) \neq P(X = a)$ but

$$P(X \in [a, a + \epsilon]) = \int_a^{a+\epsilon} f(x)dx =$$



$\approx$



$= f(a) \cdot \epsilon$

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i.e. $f(a) \approx \frac{P(X \in [a, a+\epsilon])}{\epsilon}$

Q) X = die roll, $Y \sim Unif[0,2]$ are independent

$$P(X+Y \leq 3) = ?$$

$$P(X+Y \leq 3) = P(X+Y \leq 3 \mid X=1) \cdot P(X=1)$$
$$+ P(X+Y \leq 3 \mid X=2) \cdot P(X=2)$$
$$+ P(X+Y \leq 3 \mid X=3) \cdot P(X=3)$$

$$= P(1+Y \leq 3 \cancel{\mid X=1}) \cdot 1/6$$
$$+ P(2+Y \leq 3) \cdot \frac{1}{6}$$
$$+ P(3+Y \leq 3) \cdot \frac{1}{6}$$

$$= \boxed{1 \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{6}} + 0$$

Q) $X \sim Unif[0,1]$. What is dist of $3X$?

$$P(3X \le t) = \begin{cases} 0 & \text{if } x \le 0 \\ 1 & \text{if } x \ge 3 \\ ? & x \in [0,3] \end{cases}$$

Let $t \in [0,3]$, $P(3x \le t) = P\left(X \le \frac{t}{3}\right) = \frac{t}{3} = \int_0^t \frac{1}{3}\, dt$

pdf

So, for $t > 3$, $P(3x \le t) = \int_0^3 \frac{1}{3}\, dx = 1$

$t < 0$ $P(3x < t) = \int_{-\infty}^{t} 0\, dx = 0$

Generally, $P(3x \le t) = \int_{-\infty}^{t} f(x)\, dx$

$$f(x) = \begin{cases} 0 & x \le 0 \\ \frac{1}{3} & x \in [0,3] \\ 0 & x \ge 3 \end{cases}$$