Section 2.4

Independent trials

Trial = run of an experiment

Consider experiment with two possible outcomes:

$1 :=$ success, occurs with probability p

$0 :=$ failure, occurs with probability $1 - p$.

Repeat experiment n times and assume independence of each run of experiment (e.g., flip coin n times).

$$\Omega = \{(s_1, s_2, \ldots s_n) : s_i \in \{0,1\}\},$$

All sequences of 0's & 1's of length n.

$$P\{(1,1,1)\} = p \cdot p \cdot p = p^3,$$

$$P\{(0,1,0)\} = (1-p) \cdot p (1-p) = p(1-p)^2,$$

$$P\{(0,0,1)\} = p(1-p)^2$$

$$P\{(s_1, s_2, \ldots, s_n)\} = p^{\#\text{ of 1's}} \cdot (1-p)^{\#\text{ of 0's}}$$

Three random variables (built based on independent trials)

- Bernoulli
- Binomial
- Geometric

Bernoulli r.v. (just one experiment)

Def: A Bernoulli r.v. with parameter p satisfies:

$$X = \begin{cases} 1 & \text{w.p. } p \\ 0 & \text{w.p. } 1-p \end{cases}$$

(like a single biased coin flip)

$X \sim Bern(p)$

Notation:

- w.p. = "with probability"
- $X \sim Bern(p)$ = "X is a Bernoulli r.v. with parameter p"
  ="X has the Bernoulli distribution with parameter p"
  ="The probability distribution of X is $Bern(p)$"

n trials

Q: What is the probability of k successes in n trials?

Ex) $A =$ "1 success in 3 trials" = $\{\underline{(1,0,0),\ (0,1,0),\ (0,0,1)}\}$

$$P(A) = P(1,0,0)\cdot 3 = 3\cdot p(1-p)^2$$

Generally, if A = {k successes in n trials}

= {All reorderings of $(\underbrace{1,1,\ldots 1}_{k}, 0, 0, \ldots 0)$}

choose k places to put a 1

$$\underbrace{\_\ \underline{1}\ \_\ \_\ \underline{1}\ \underline{1}}_{n}$$

$$P(A) = \frac{P\{(\overbrace{1,1,1\ldots 1}^{k}, 0, \ldots 0)\}\cdot\binom{n}{k}}{}$$

$$= \binom{n}{k}\cdot p^k(1-p)^{n-k}$$

Binomial r.v. (n trials, probability of success p)

X = # of successes in n trials

Def: A binomial r.v. with parameters n and p satisfies

$$\underbrace{P(X = k) = \binom{n}{k} \cdot p^k (1-p)^{n-k}}$$ $k \in \{0, 1, \ldots n\}$

Recall, this is the prob. mass fcn (pmf).

Notation: $X \sim Bin(n, p)$

Ex) Roll three dice. X = number of 1s rolled.

$n =$ 3, $p =$ 1/6

$P(X=0) = \binom{3}{0}\left(\frac{1}{6}\right)^0 \cdot \left(1-\frac{1}{6}\right)^3$, $P(X=1) = \binom{3}{1} \cdot \left(\frac{1}{6}\right)^1 \left(1-\frac{1}{6}\right)^2$

$= \left(\frac{5}{6}\right)^3$

$= \frac{3!}{2! \cdot 1!} \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2 = 3 \cdot \frac{1}{6} \cdot \left(\frac{5}{6}\right)^2$

$P(X=2) = \binom{3}{2} \cdot \left(\frac{1}{6}\right)^2 \cdot \frac{5}{6}$

$P(X=3) = \binom{3}{3} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^0$

Relationship between $Bern(p)$ and $Bin(n,p)$

Let $X_1, X_2, \ldots, X_n \sim Bern(p)$ and $X_1, \ldots X_n$ are indep.

Then $X_1 + X_2 + \cdots + X_n \sim$ $Bin(n,p)$

Unbounded trials

Q: Let X be # of trials until (and including) 1st success.

$\{X = k\}$ if we have $k-1$ failures followed by 1 success

$P(X = k) =$ P(k-1 failures) · P(1 success)

Def: A geometric r.v. with parameter p satisfies:

$P(X = k) = (1-p)^{k-1} \cdot p$, $\quad k = 1, 2, 3, \ldots$

Notation: $X \sim Geom(p)$

Q) Roll two dice (at same time) until you get "snake eyes" i.e., both 1s. What is the probability you roll both dice at least 17 times?

X = # of die rolls until snake eyes

$X \sim \text{Geom}(p)$, $p = P(\text{snake eyes}) = P((1,1)) = 1/36$

$$P(\text{roll} \geq 17 \text{ times}) = P(X \geq 17) = \sum_{k=17}^{\infty} P(X=k)$$

$$= \sum_{k=17}^{\infty} (1-p)^{k-1} \cdot p \underset{\substack{\uparrow \\ \text{sum geom. series}}}{=} p\,(1-p)^{17-1} \cdot \underbrace{\frac{1}{1-(1-p)}}_{1/p} = (1-p)^{16}$$

$$= \left(1 - \frac{1}{36}\right)^{16}$$

Alternative soln:

$$P(\text{roll} \geq 17 \text{ times}) = P(\text{first 16 rolls "fail"}) = (1-p)^{16} = \left(1-\frac{1}{36}\right)^{16}$$

Q) Again, roll two dice (at same time). "success" = the sum is even. Roll until "success". P("you roll both dice an even number of times")=?

X = # of rolls until success. $X \sim Geom(p)$

$p = P(\text{sum is even}) = 1/2$

exercize

$P(\text{even \# of rolls}) = P(X \text{ is even})$

$= P(X=2) + P(X=4) + P(X=6) + \ldots$

$= (1-p)\cdot p + (1-p)^3 \cdot p + (1-p)^5 p + \ldots$

$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \ldots$

$= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \ldots = \frac{1}{4} \frac{1}{1-\frac{1}{4}} = \frac{1}{4} \cdot \frac{1}{3/4} = \boxed{\frac{1}{3}}$