- HW 1 available. Due next Friday.
- Practice problems for Chapters 1 & 2 available in Canvas quizzes.

Section 2.3

Independence

$P(A|B) =$ the probability that A occurs given B occurred

Independence: B has nothing to do with A! (Or at least it doesn't effect the probability.) I.e., $P(A) = P(\, A \mid B\,) = \frac{P(AB)}{P(B)}$.

Examples?

Def: Events A and B are *independent* iff

$$P(AB) = P(A) \cdot P(B)$$

Ex) Roll two dice.

$$A = \{first\ die = 1\}, \qquad B = \{second\ die = 1\},$$
$$C = \{sum\ of\ dice = 7\}, \qquad D = \{sum\ of\ dice = 3\}$$

Q) Which pairs of events are independent?

$\underline{A \,\&\, B}$

$P(AB) = P(\text{both dice} = 1) = P\{(1,1)\} = \frac{1}{36}$

$P(A)\cdot P(B) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$

$\Big\}$ $P(AB) = P(A)\cdot P(B)$ so A & B are independent.

$\underline{A \,\&\, C}$

$P(AC) = P(\text{first die} = 1,\ \text{sum} = 7)$

$= P(\text{first die} = 1,\ \text{second} = 6)$

$= P\{(1,6)\} = \frac{1}{36}$

$P(A) = \frac{1}{6}$, $P(C) = P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = \frac{6}{36} = \frac{1}{6}$

so $P(A)\cdot P(C) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36} = P(AC)$ i.e. A & C are independent.

C & D

$C \cap D = \{\}$, $P(CD) = 0 \neq P(C)\cdot P(D)$ so C & D are dependent

Disjoint $\neq$ independent in general

A & D

$P(AD) = P(\text{1st die} = 1, \text{sum} = 3) = P\{(1,2)\} = \frac{1}{36}$

$P(A) = \frac{1}{6}$, $P(D) = P\{(1,2),(2,1)\} = \frac{2}{36}$ so $P(A)\cdot P(D) = \frac{1}{6}\cdot\frac{2}{36} \neq P(AD)$

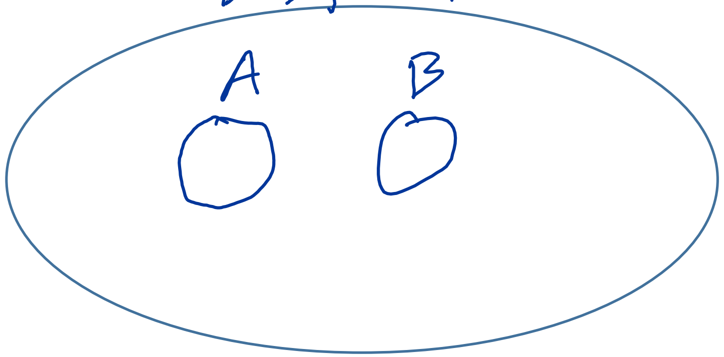
so dependent

A few points about independence:

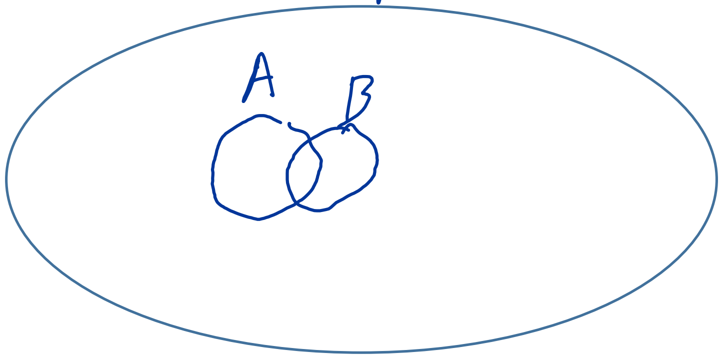
- If A and B are independent, then so are A and B^c, also independent

- Mutually exclusive (i.e. disjoint) $\neq$ independent

Disjoint



Independent



If $P(A), P(B) > 0$, & $A \cap B = \{\}$, then

$P(AB) = P\{\} = 0$

$\neq \underbrace{P(A) \cdot P(B)}_{>0}$

Special case: If $P(A)$ or $P(B) = 0$, then $P(AB) = 0 = P(A) \cdot P(B)$

so A & B will be indep even if they are disjoint.

Independence picture: Proportional overlap

Ex) Roll a die with 100 sides. $A = \{roll\ is\ in\ \{1,2,3,\dots 10\}\}$. $B = \{roll\ is\ in\ \{10, 20, 30, \dots 100\}\}$.
Are A and B independent?

$P(AB) = P\{10\} = \frac{1}{100}$

$P(A) \cdot P(B) = \frac{10}{100} \cdot \frac{10}{100} = \frac{1}{100}$

so A & B are independent, so

$$\underbrace{P(A|B)}_{\frac{P(AB)}{P(B)}} = \underbrace{P(A)}_{\frac{P(A)}{P(\Omega)}}$$

Ω

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Independence with more than two events

Def: Events $A_1, A_2, \dots A_n$ are *independent* iff for any set of indices $1 \le i_1 < i_2 < \cdots < i_k \le n$ we have

$$P\left(A_{i_1} A_{i_2} \cdots A_{i_k}\right) = P\left(A_{i_1}\right) \cdot P\left(A_{i_2}\right) \cdots P\left(A_{i_k}\right).$$

E.g., 3 events A, B, C are independent if:

$$\left.\begin{aligned} P(AB) &= P(A)\cdot P(B) \\ P(AC) &= P(A)\cdot P(C) \\ P(BC) &= P(B)\cdot P(C) \end{aligned}\right\} \text{ Called "pairwise independence"}$$

$$P(ABC) = P(A)\cdot P(B)\cdot P(C)$$

Pairwise independence vs independence

Ex) Roll two dice.

$$A = \{first\ die = 1\}, \qquad B = \{second\ die = 1\},$$
$$C = \{sum\ of\ dice = 7\}$$

Are A, B, C independent?

A, B are indep. as above
A, C " " " "
B, C " " " "

$P(ABC) \stackrel{?}{=} P(A) \cdot P(B) \cdot P(C)$

LHS: $\underbrace{P(ABC)}_{\{\text{first die}=1,\ \text{second die}=1,\ \text{sum}=7\} = \{\}} = 0 \neq \underbrace{P(A)}_{1/6} \cdot \underbrace{P(B)}_{1/6} \cdot \underbrace{P(C)}_{1/6}$

A, B, C are "pairwise indep.", but they are not indep.

Independent random variables

Def: Let $X_1, X_2, \dots X_n$ be random variables defined on the same probability space. Then $X_1, X_2, \dots, X_n$ are independent iff

$$P(X_1 \in B_1, X_2 \in B_2, \dots, X_n \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i)$$

for all choices of subsets $B_1, B_2, \dots, B_n$ of the real line.*

Ex) X, Y are indep if

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B) \quad \text{for any } A, B \subseteq \mathbb{R}$$

Ex) X, Y are indep. die rolls, then

$$P(X \in [3,5], Y \in [4,7]) = P(X \in [3,5]) \cdot P(Y \in [4,7])$$

Independent *discrete* random variables

Fact: Let $X_1, X_2, \ldots X_n$ be **discrete** random variables defined on the same probability space. Then $X_1, X_2, \ldots, X_n$ are independent iff

$$P(X_1 = k_1, X_2 = k_2, \ldots, X_n = k_n) = \Pi_i P(X_i = k_i)$$

for all choices of $k_1, k_2, \ldots, k_n$ *(of values that r.v.'s can take).*

Ex) Roll two dice. $X = first\, roll,\ Y = second\, roll$

$$P(X=i, Y=j) = P(X=i)\cdot P(Y=j) \qquad i, j \in \{1, 2, \ldots 6\}$$

$$= 1/6 \cdot 1/6$$

Ex) Roll two dice. $X = first\ roll, Y = sum\ of\ rolls$.

Are X and Y independent?

$$P(X = k_1, Y = k_2) \stackrel{?}{=} P(X = k_1) \cdot P(Y = k_2)$$

for $k_1 = \{1, 2, 3, \dots 6\}$, $k_2 = \{2, 3, \dots, 12\}$.

$k_2 = 2$, $k_1 = 6$

$$\left.\begin{array}{l} P(X=6, Y=2) = 0 \\ P(X=6) \cdot P(Y=2) = \frac{1}{6} \cdot \frac{1}{36} \neq 0 \end{array}\right\}$$

Thus, X & Y are dependent.

Ex) X = 1st die roll. $Y = \begin{cases} 1 & \text{if sum of die rolls} = 7 \\ 0 & \text{else} \end{cases}$

Let $k \in \{1, 2, \dots 6\}$

$$P(X=k, Y=1) = P(\text{1st die} = k, \text{ sum} = 7)$$
$$= P(\text{1st die} = k, \text{ second die} = 7-k)$$
$$= P(\text{1st die} = k) \cdot P(\text{second die} = 7-k) \quad \text{by indep.}$$
$$= 1/6 \cdot 1/6$$

$$P(X=k) \cdot P(Y=1) = \frac{1}{6} \cdot P(\text{sum} = 7) = \frac{1}{6} \cdot \frac{1}{6}$$

So $P(X=k, Y=1) = P(X=k) \cdot P(Y=1)$ $\quad k \in \{1, 2, \dots 6\}$

$\Rightarrow P(X=k, Y=0) = P(X=k) \cdot P(Y=0)$ since $\{Y=0\} = \{Y=1\}^c$

so we can use indep of $\{X=k\}, \{Y=1\}$ to imply indep. of $\{X=k\}, \{Y=0\}$