Section 2.3

Independence

$P(A|B) =$ the probability that A occurs given B occurred

Independence: B has nothing to do with A! (Or at least it doesn't effect the probability.) I.e., $P(A) = P(\, A \mid B \,) = \frac{P(AB)}{P(B)}$.

Examples?

Def: Events A and B are *independent* iff

$$P(AB) = P(A) \cdot P(B)$$

Ex) Roll two dice.

$$A = \{first\ die = 1\}, \qquad B = \{second\ die = 1\},$$
$$C = \{sum\ of\ dice = 7\}, \qquad \mathrm{D} = \{sum\ of\ dice = 3\}$$

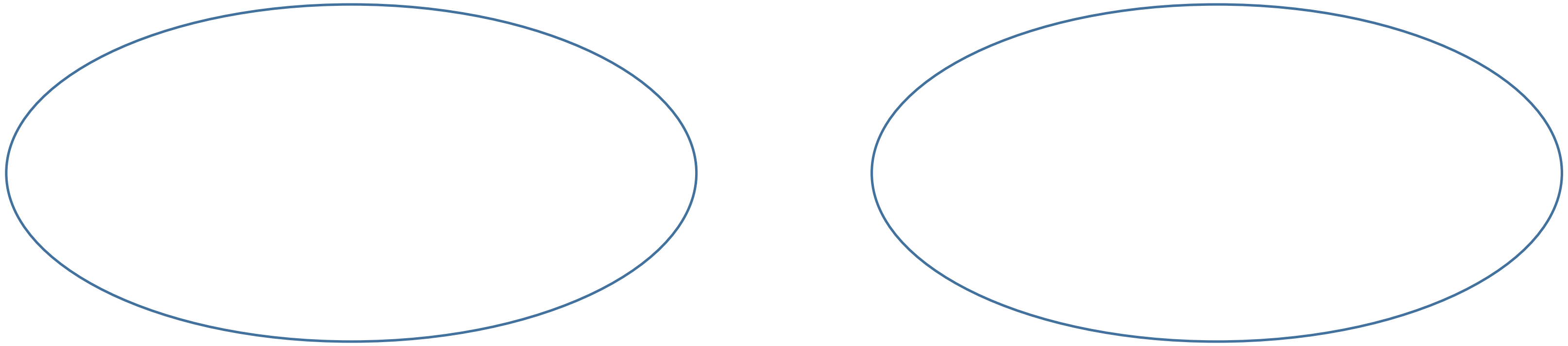
Q) Which pairs of events are independent?

A few points about independence:

- If A and B are independent, then so are A and B^c,

also ______________________________

- Mutually exclusive (i.e. disjoint) $\neq$ independent



Independence picture: Proportional overlap

Ex) Roll a die with 100 sides. $A = \{roll\ is\ in\ \{1,2,3, \ldots 10\}\}$. $B = \{roll\ is\ in\ \{10, 20, 30, \ldots 100\}\}$. Are A and B independent?

$P(AB) =$ ________________

$P(A) \cdot P(B) =$ ________________

Ω

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Independence with more than two events

Def: Events $A_1, A_2, \ldots A_n$ are *independent* iff for any set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ we have

$$P\left(A_{i_1} A_{i_2} \cdots A_{i_k}\right) = P\left(A_{i_1}\right) \cdot P\left(A_{i_2}\right) \cdots P\left(A_{i_k}\right).$$

E.g., 3 events A, B, C are independent if:

Pairwise independence vs independence

Ex) Roll two dice.

$$A = \{first\ die = 1\}, \qquad B = \{second\ die = 1\},$$
$$C = \{sum\ of\ dice = 7\}$$

Are A, B, C independent?

Independent random variables

Def: Let $X_1, X_2, \ldots X_n$ be random variables defined on the same probability space. Then $X_1, X_2, \ldots, X_n$ are independent iff

$$P(X_1 \in B_1, X_2 \in B_2, \ldots, X_n \in B_n) = \prod_{i=1}^{n} P(X_i \in B_i)$$

for all choices of subsets* $B_1, B_2, \ldots, B_n$ *of the real line.*

Independent *discrete* random variables

Fact: Let $X_1, X_2, \dots X_n$ be **discrete** random variables defined on the same probability space. Then $X_1, X_2, \dots, X_n$ are independent iff

$$P(X_1 = k_1, X_2 = k_2, \dots, X_n = k_n) = \Pi_i P(X_i = k_i)$$

for all choices of $k_1, k_2, \dots, k_n$ *(of values that r.v.'s can take).*

Ex) Roll two dice. $X = first\, roll,\ Y = second\, roll$

Ex) Roll two dice. $X = first\ roll, Y = sum\ of\ rolls$.

Are X and Y independent?

$$P(X = k_1, Y = k_2) \stackrel{?}{=} P(X = k_1) \cdot P(Y = k_2)$$

for $k_1 = \{1, 2, 3, \ldots 6\}$, $k_2 = \{2, 3, \ldots, 12\}$.