Section 2.2

Bayes' Formula

Again: Flip a coin three times. For each head, roll a die. Then sum the die rolls.

$P(\,sum = 3 \mid 1\,H\,) = $ ______ $P($1 die roll $= 3) = 1/6$

$P(1\,H|sum = 3) = ?$

Bayes' Formula:

$$P(B \mid A) = \frac{P(A \mid B) \cdot P(B)}{P(A)} = \frac{P(A \mid B) \cdot P(B)}{P(A \mid B) \cdot P(B) + P(A \mid B^c) \cdot P(B^c)}$$

So $P(1\ H | sum = 3) = \frac{P(sum=3 \mid 1\ H) \cdot P(1\ H)}{P(sum=3)} = \frac{1/6 \cdot 3/8}{\underbrace{P(sum=3)}_{\text{last lecture}}}$

Proof of Bayes' formula:

$$P(B|A) = \frac{P(BA)}{P(A)} = \frac{P(AB)}{P(A)} = \frac{P(A|B) \cdot P(B)}{P(A)} \quad \checkmark$$

Rare disease paradox

Ex) Suppose a rare disease effects $1/10^6$ of the population. You randomly decide to get tested, and you test positive! What is the probability you have the disease?

$$D = \{you\ have\ disease\}$$
$$T = \{you\ test\ positive\}$$

The test is very accurate:

- Proportion of false positives = $\frac{1}{10^4} = P(T|D^c)$
- Proportion of false negatives = $\frac{1}{10^5} = P(\underline{T^c|D})$

$P(T|D) = 1 - P(T^c|D)$
$= 1 - 1/10^5$

$P(you\ have\ disease\ |you\ test\ positive) =$ $\underline{P(D|T)}$

$$P(D|T) = \frac{P(T|D)\cdot P(D)}{P(T)} = \frac{P(T|D)\cdot P(D)}{P(T|D)\cdot P(D) + P(T|D^c)\cdot P(D^c)}$$

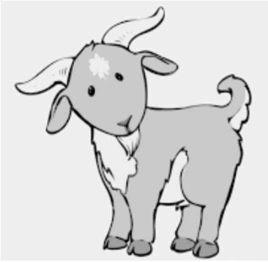
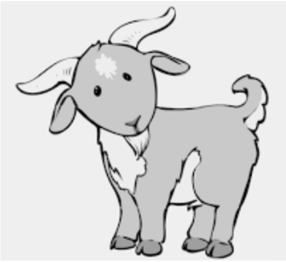
Bayes Formula

$$= \frac{(1-10^{-5})\,10^{-6}}{(1-10^{-5})\cdot 10^{-6} + 10^{-4}\cdot(1-10^{-6})}$$

$$\approx \frac{10^{-6}}{10^{-6}+10^{-4}} = \frac{1}{1+100} = \frac{1}{101}$$

Monty Hall Problem

Door 1 Door 2 Door 3

Should you switch doors.

$C = \{$the door you first picked has a car$\}$

$G = \{$A goat is revealed$\}$

$P(C \mid G) = ?$ Depends on host's strategy:

Good strategy: whatever door you pick, I show a goat.

$$P(C|G) = \frac{P(G|C) \cdot P(C)}{P(G)} = \frac{1 \cdot 1/3}{1} = 1/3$$

Thus 2/3 chance you win if you switch doors.

Evil strategy: I only show a goat if you pick the car door.

$$P(C|G) = \frac{P(G|C) \cdot P(C)}{\underbrace{P(G)}_{P(C)}} = \frac{1 \cdot 1/3}{1/3} = 1$$