Chapter 2

Conditional probability and independence

Section 2.1

Conditional probability

Ex) Let x be result of a die roll

Pmf: $p(k) = P\{x = k\} = \underline{\ 1/6\ }$ $k = 1, 2, \ldots, 6$

Now I tell you "$x \leq 4$".

$P\{x = 5 \text{ given } x \leq 4\} = \underline{\ 0\ }$, $P\{x = 6 \text{ given } x \leq 4\} = \underline{\ 0\ }$

$P\{x = k \text{ given } x \leq 4\} = \underline{\ 1/4\ }$ for $k = 1,2,3,4$.

Conditioning on an event

Q: In general, we can *condition* on an event $B \overset{\subseteq}{\cancel{\in}} \Omega$. Conditioning updates all probabilities (new probability measure). What is the new probability measure?

“given”

We write $P(\,A \mid B\,)$ = “probability that A occurs given that B occurred”

So, $$\begin{aligned} P(A \cap B) &= P(\text{"A and B occur“}) \\ &= P(\text{"B occurs"}) \cdot P(\text{"}A \textit{ occurs given } B\text{“}) \\ &= P(B) \cdot P(A \mid B) \end{aligned}$$

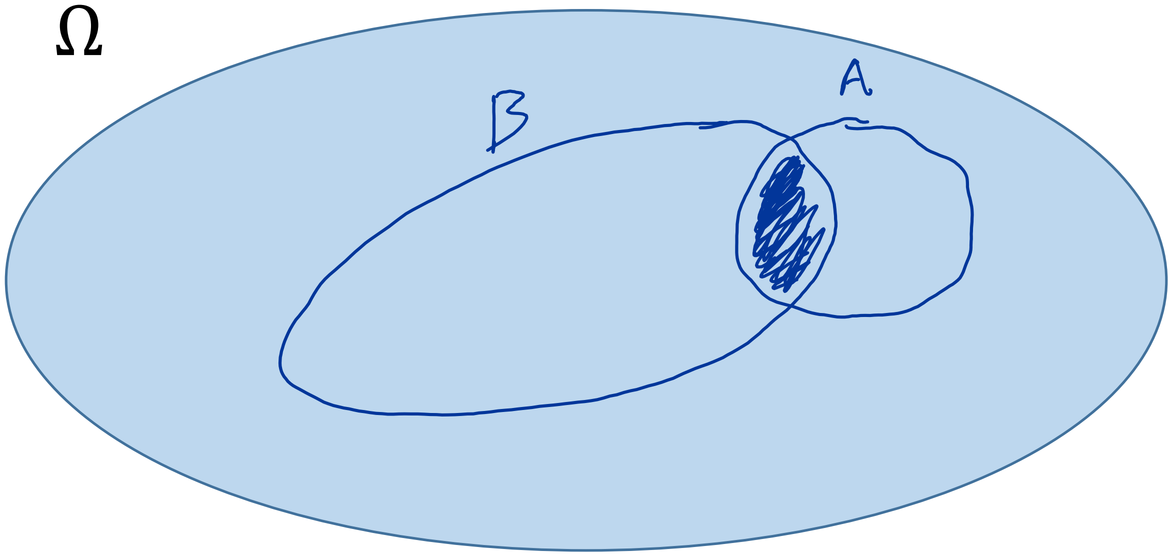
Def: Conditional probability

$$P(\,A \mid B\,) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

Alternative interpretation

When we condition on B, B becomes the new sample space. Divide by $P(B)$ to normalize, so that $P(\,B \mid B\,) = 1$. The collection $\{P(\,A \mid B\,)\}_{A\in\Omega}$ is an updated probability measure.

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$



Ex) Roll two dice, x = sum.

$$P(x = 5 \mid first\ die\ is\ 3) = \underline{P(\text{second die} = 2) = 1/6}$$

$$P(x=5 \mid \text{first die is } 3) = \frac{P(x=5 \ \& \ \text{first die is } 3)}{P(\text{first die is } 3)}$$

$$= \frac{P(\text{first die is } 3 \ \& \ \text{second die is } 2)}{P(\text{first die is } 3)}$$

$$= \frac{1/36}{1/6} = 1/6$$

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

$$P(\text{first die is } 3 \mid x=5)$$

$$= \frac{P(\text{first die is } 3,\ x=5)}{P(x=5)}$$

$$= \frac{1/36 \leftarrow \text{as above}}{P\{(1,4),(2,3),(3,2),(4,1)\}} = \frac{1/36}{4/36} = 1/4$$

Ex) An urn has 9 red balls and 7 green balls. Draw 2 at random.

$$P\{both\ green\} = P(G1\,G2) = \underbrace{\frac{7}{16}}_{P(G_1)} \cdot \underbrace{\frac{6}{15}}_{P(G_2|G_1)}$$

Now 4 drawn:

$$P(G_1R_2R_3G_4) = \underbrace{\frac{7}{16}}_{P(G_1)} \cdot \underbrace{\frac{9}{15}}_{P(R_2|G_1)} \cdot \underbrace{\frac{8}{14}}_{P(R_3|G_1R_2)} \cdot \underbrace{\frac{6}{13}}_{P(G_4|G_1R_2R_3)}$$

Fact:

$$P(A_1A_2A_3 \cdots A_n) = P(A_1) \cdot P(A_2 \mid A_1) \cdot P(A_3|A_1A_2) \cdots P(A_n \mid A_1A_2 \ldots A_{n-1})$$

Ex) An urn has 9 red balls and 7 green balls. Two drawn.

$P(R_2) =$ $P(R_1) = \boxed{9/16}$

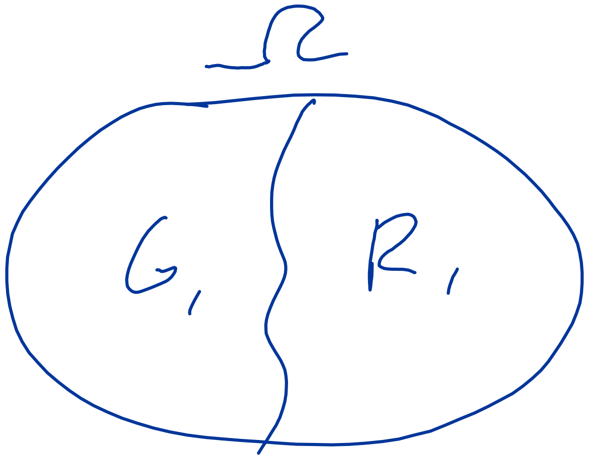
$$P(R_2) = P(\underbrace{(G_1 \cup R_1)}_{\Omega} \cap R_2) = P((G_1 \cap R_2) \cup (R_1 \cap R_2))$$

$$= P(G_1 \cap R_2) + P(R_1 \cap R_2)$$

$$= P(G_1) \cdot P(R_2 | G_1) + P(R_1)\, P(R_2 | R_1)$$

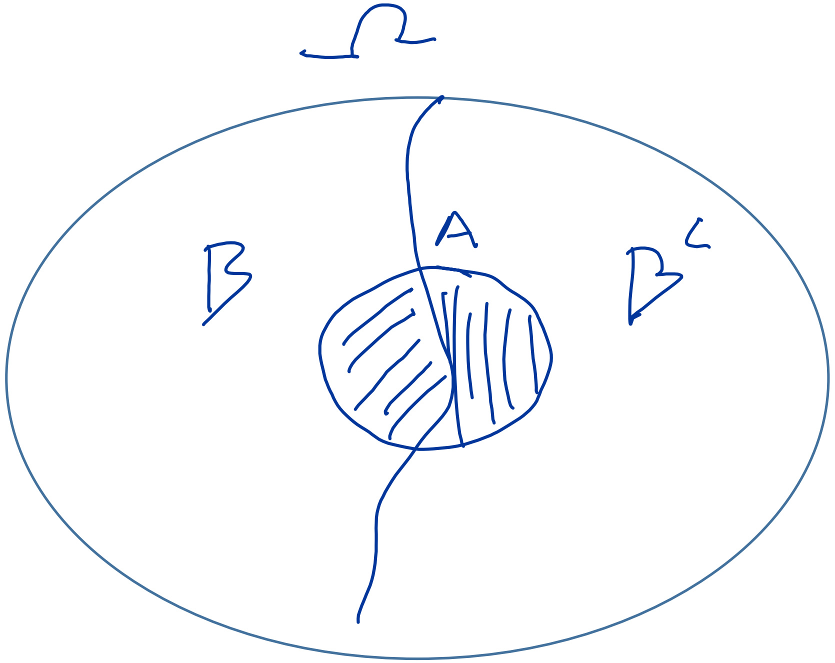
$$= \frac{7}{16} \cdot \frac{\cancel{9}^{\,3}}{\cancel{15}_{\,5}} + \frac{\cancel{9}^{\,3}}{16} \cdot \frac{8}{\cancel{15}_{\,5}} = \frac{7 \cdot 3}{16 \cdot 5} + \frac{3 \cdot 8}{16 \cdot 5}$$

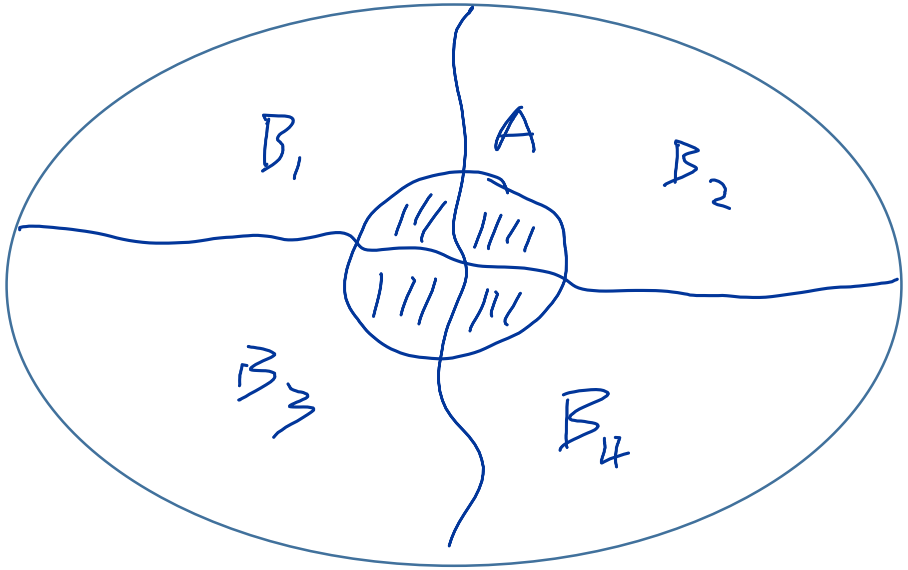
3/5



$$= \frac{21 + 24}{16 \cdot 5} = \frac{45}{16 \cdot 5} = \boxed{\frac{9}{16}}$$

Fact: $P(A) = P(AB) + P(AB^c)$
$$= P(A \mid B) \cdot P(B) + P(A \mid B^c) \cdot P(B^c)$$





Fact: If $B_1, B_2, \ldots, B_n$ is a partition of Ω, then

$$P(A) = \sum_i P(A\,B_i) = \sum_i P(A \mid B_i) \cdot P(B_i)$$

Ex) Flip a coin three times. For each head, roll a die. What is $P\{Sum\ of\ all\ dice\ rolled = 3\}$?

$X = \#$ of heads

$Y =$ sum of dice

$$P(Y=3) = P(Y=3 \mid X=0) \cdot P(X=0)$$
$$+ P(Y=3 \mid X=1) \cdot P(X=1)$$
$$+ P(Y=3 \mid X=2)\ P(X=2)$$
$$+ P(Y=3 \mid X=3) \cdot P(X=3)$$

$P(X=0) = P\{TTT\} = 1/8$

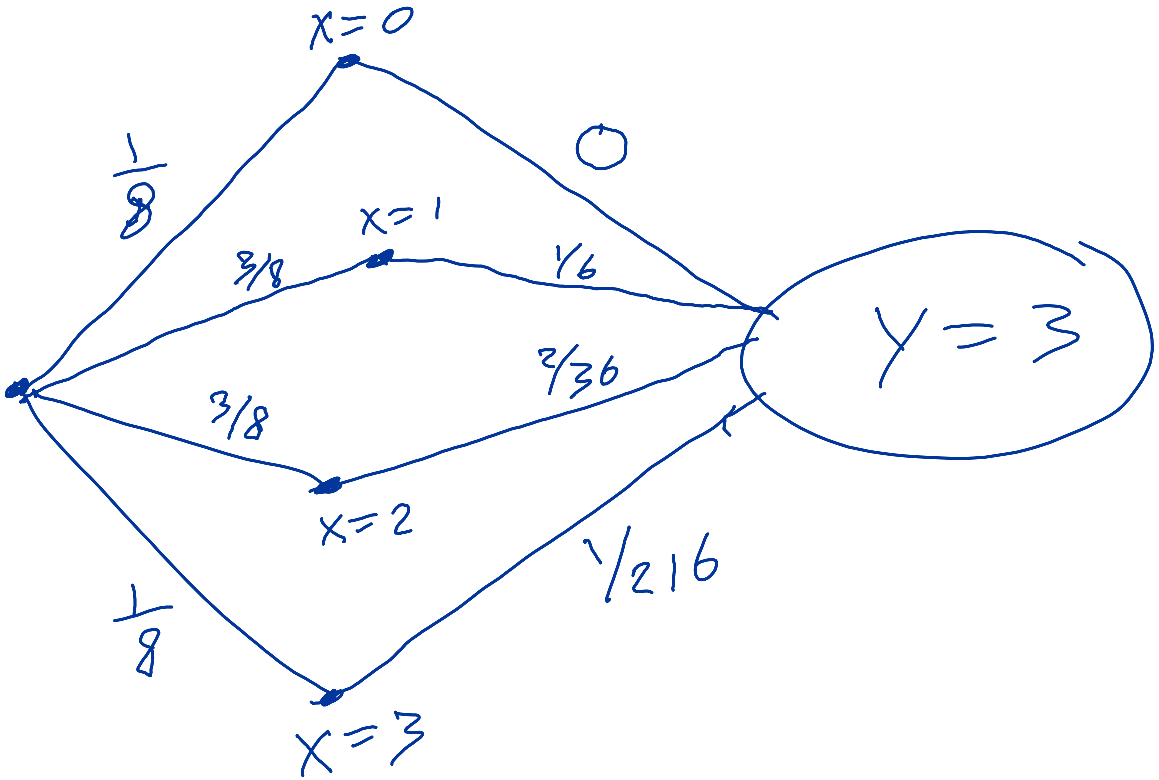
$P(X=1) = P\{HTT, THT, TTH\}$

$= 3/8$

$P(X=2) = 3/8$

$P(X=3) = 1/8$

$$\boxed{\begin{aligned} = \quad & 0 \\ + \quad & 1/6 \cdot 3/8 \\ + \quad & 2/36 \cdot 3/8 \\ + \quad & 1/216 \cdot 1/8 \end{aligned}}$$


x=0
1/8
x=1
3/8
0
1/6
3/8
2/36
x=2
1/8
x=3
1/216
Y=3