

# Section 1.5

Random variables: a first look

Ex) Roll two dice, let  $x$  be the sum

$x$  is a random variable. What is its relationship to the underlying sample space?

$$\Omega = \{(1,1), (1,2), (1,3), \dots, (6,5), (6,6)\}$$

$x$  outputs a number for each outcome, i.e.,  $x$  is a *function*.  $x: \Omega \rightarrow \mathbb{R}$ .

$$x((1,1)) = 2, \quad x((3,5)) = \underline{8}, \quad x((i,j)) = \underline{i+j}$$

**Definition:** A random variable (r.v.) is a function from  $\Omega$  to  $\mathbb{R}$ .

Ex) Roll two dice, let  $x$  be the sum

What is  $P\{x = 3\}$ ?

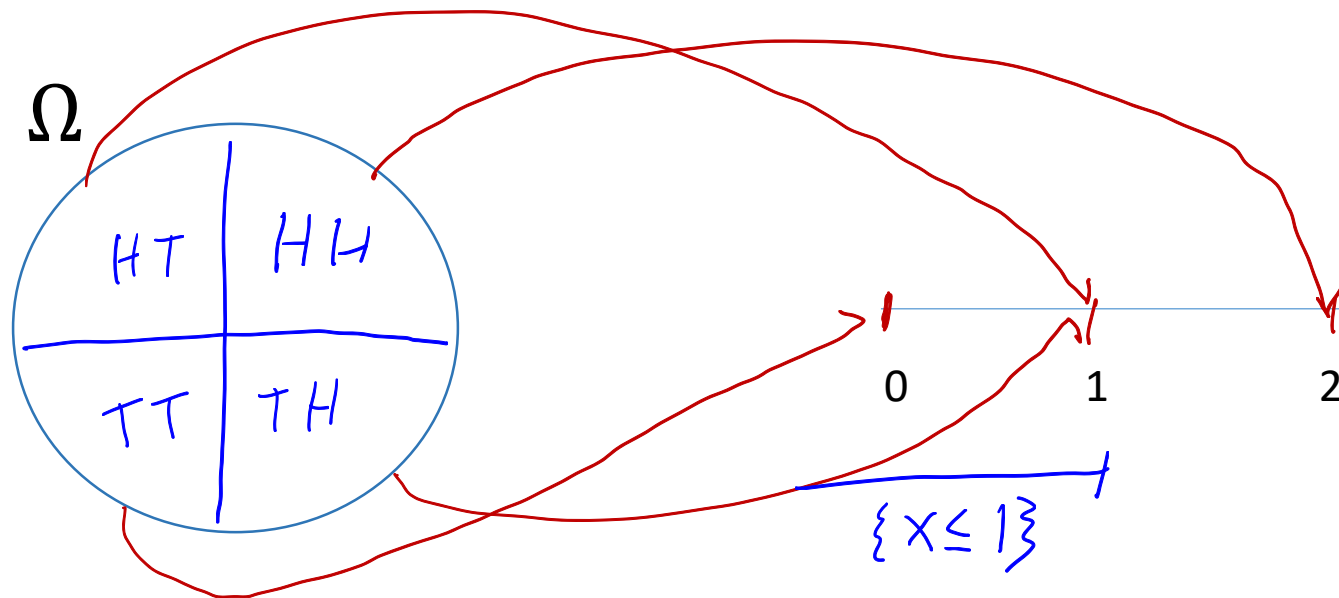
$\{x = 3\}$  is shorthand for  $\{\omega \in \Omega: x(\omega) = 3\}$ . This is the *preimage* of 3.

$$\{x = 3\} = \{(1,2), \underline{(2,1)}\}$$

$$P\{x = 3\} = P\{\underline{(1,2)}, \underline{(2,1)}\} = \underline{\frac{2}{36}}$$

Ex) Flip two coins.  $x$  = number of heads

Which outcomes correspond with  $x \leq 1$ ?



$$P\{x \leq 1\} = P\{\omega: x(\omega) \leq 1\} = P\{TT, \underline{TH}, \underline{HT}\} = \underline{\underline{\frac{3}{4}}}$$

# Probability distribution

Goal: completely specify the “randomness” of r.v.  $x$ .

Definition (def): The *probability distribution* of a r.v.  $x$  is the collection of probabilities  $P\{x \in B\}$  for sets  $B \subset \mathbb{R}$ .

$$\{x \in B\} = \text{preimage}(B) = \{\omega \in \Omega: x(\omega) \in B\}$$

# Discrete r.v.'s

Both random variables above were *discrete*, i.e, they take values on a discrete set, i.e., they only take at most a countably infinite number of values.

In discrete case, the probability distribution is captured by:

Def: The *probability mass function (pmf)* of a discrete r.v.  $x$  is the collection of probabilities  $p(k) = P\{x = k\}$  for all values  $k$  that  $x$  may take.

Relation to probability dist:  $P(x \in B) = \frac{\sum_{k \in B} P\{x=k\}}{\sum_{k \in B} P(k)}$

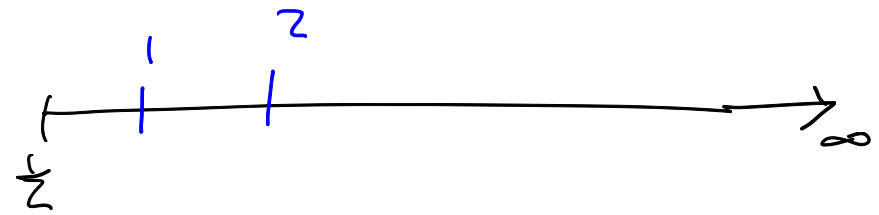
Ex)  $x$  = number of heads in 2 flips

pmf of  $x$ ?  $x$  can take the values: 0, 1, 2

$$p(0) = \frac{P\{TT\}}{=} \frac{1}{4}, \quad p(1) = \frac{P\{TH, HT\}}{=} \frac{1}{2}, \quad p(2) = \frac{P\{HH\}}{=} \frac{1}{4}$$

pmf of  $x$ .

$$P\left(X \in \left(\frac{1}{2}, \infty\right)\right) = P(1) + P(2) = \frac{1}{2} + \frac{1}{4} = \boxed{\frac{3}{4}}$$



Q: Consider probability space defined by choosing an outcome uniformly at random on  $[0,1]$ . Can you make a discrete r.v. on this space? Define the mapping  $X: [0,1] \rightarrow \mathbb{R}$ ,

$$X(\omega) = \begin{cases} 0 & \omega \in [0, \frac{1}{2}) \\ 1 & \omega \in [\frac{1}{2}, 1] \end{cases}$$

$X$  takes values 0 or 1 i.e.

a finite # of values, so  $X$  is discrete.

pmf of  $X$ :  $P(0) = P\{X=0\} = P([0, \frac{1}{2})) = \frac{1}{2}$

$$P(1) = P\{X=1\} = P([\frac{1}{2}, 1]) = \frac{1}{2}.$$