Section 1.5

Random variables: a first look

Ex) Roll two dice, let x be the sum

x is a random variable. What is its relationship to the underlying sample space?

$$\Omega = \{(1,1), (1,2), (1,3), \ldots (6,5), (6,6)\}$$

x outputs a number for each outcome, i.e., x is a *function*. $\underline{x : \Omega} \to \mathbb{R}$.

$$x\big((1,1)\big) = 2, \qquad x\big((3,5)\big) = \underline{\ 8\ }, \qquad x\big((i,j)\big) = \underline{\ i+j\ }$$

Definition: A random variable (r.v.) is a function from Ω to $\mathbb{R}$.

Ex) Roll two dice, let x be the sum

What is $P\{x = 3\}$?

$\{x = 3\}$ is shorthand for $\{\underset{\sim}{\omega} \in \underset{\sim}{\Omega} : x(\omega) = 3\}$. This is the *preimage* of 3.

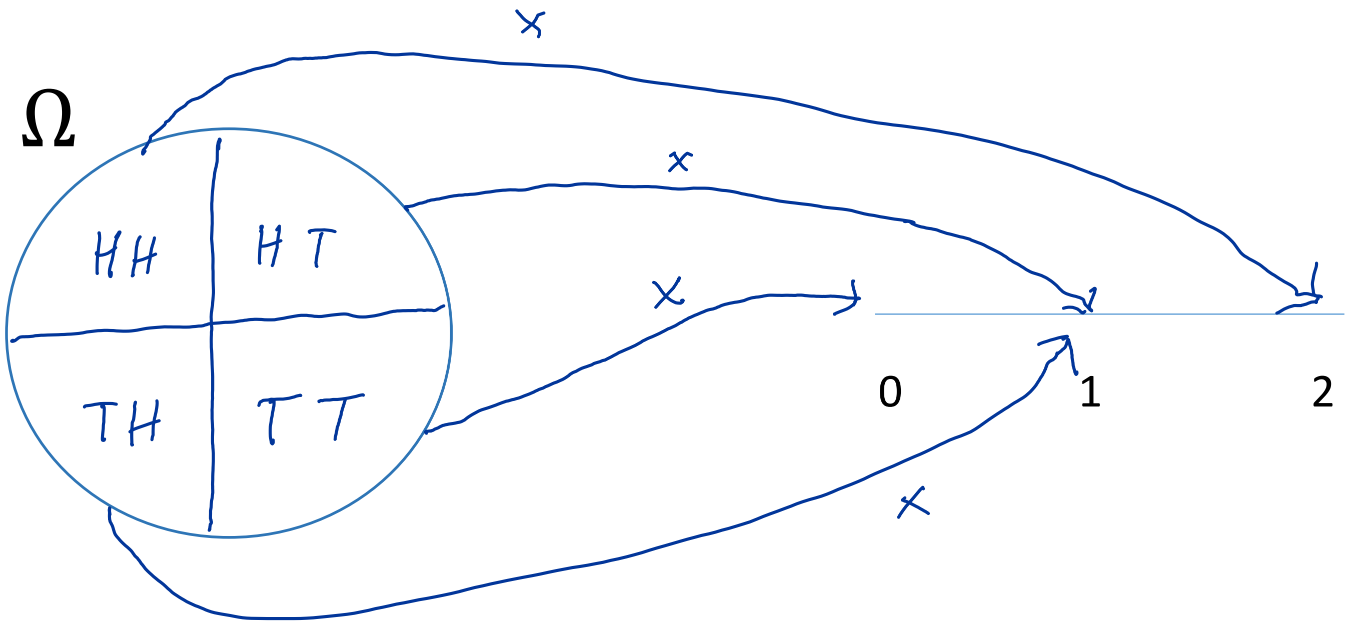
$$\{x = 3\} = \{(1,2), \ \underline{(2,1)}\ \}$$

$$P\{x = 3\} = P\{\underline{(1,2),\ (2,1)}\} = \frac{\# A}{\# \Omega} = \frac{2}{36}$$

$$P\{x=4\} = P\{(1,3), (2,2), (3,1)\} = \frac{3}{36}$$

Ex) Flip two coins. x = number of heads

Which outcomes correspond with $x \leq 1$?



$$P\{x \leq 1\} = P\{\omega: x(\omega) \leq 1\} = P\{TT, \underline{\ TH\ }, \underline{\ HT\ }\} = \underline{\ \tfrac{3}{4}\ }$$

Probability distribution

Goal: completely specify the "randomness" of r.v. x.

Definition (def): The *probability distribution* of a r.v. x is the collection of probabilities $P\{x \in B\}$ for sets $B \subset \mathbb{R}$.

$$\{x \in B\} = preimage(B) = \{\omega \in \Omega : x(\omega) \in B\}$$

Discrete r.v.'s

Both random variables above were *discrete*, i.e, they take values on a discrete set, i.e., they only take at most a countably infinite number of values.

In discrete case, the probability distribution is more simply captured by:

Def: The *probability mass function (pmf)* of a discrete r.v. x is the collection of probabilities $\mathrm{p}(k) = P\{x = k\}$ for all values k that x may take.

Relation to probability dist: $P(x \in B) = \sum_{k \in B} p(k)$

Ex) x = number of heads in 2 flips

pmf of x? x can take the values: 0, 1, 2

$$\underbrace{p(0) = \tfrac{1}{4}, \quad p(1) = \tfrac{1}{2}, \quad p(2) = \tfrac{1}{4}.}_{\text{pmf of } x}$$

$P(x \le 1.5) = p(0) + p(1) = \frac{1}{4} + \frac{1}{2}$

Q: Consider probability space defined by choosing an outcome uniformly at random on $[0,1]$. Can you make a discrete r.v. on this space?

$X(\omega) = \omega$ is continuous

$$X(\omega) = \lceil \omega \rfloor = \begin{cases} 1 & \omega \geq 1/2 \\ 0 & \omega < 1/2 \end{cases}$$

$$P(X=0) = P\{\omega \in \Omega : \omega < \tfrac{1}{2}\} = P\{[0, \tfrac{1}{2})\} = 1/2$$

$$P(X=1) = P\{[1/2, 1]\} = 1/2$$

Note: $\cdot\, P\{\tfrac{1}{2}\} = 0$, $\quad \cdot\, P([0, \tfrac{1}{2}]) = P([0, \tfrac{1}{2}) \cup \{\tfrac{1}{2}\})$

$$= P([0, \tfrac{1}{2})) + \underbrace{\cancel{P\{\tfrac{1}{2}\}}}_{0}$$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \text{ is rational} \\ 0 & \text{if not} \end{cases}$$

$$P(1) = 0 \qquad P(0) = 1$$

$$P(1) = P(\omega \text{ is rational}) = P\Big(\underbrace{\bigcup_{\omega \in \mathbb{Q}}}_{\text{rationals}} \{\omega\}\Big) = \sum_{\omega \in \mathbb{Q}} \cancelto{0}{P\{\omega\}} = 0$$

since rationals are countably infinite.