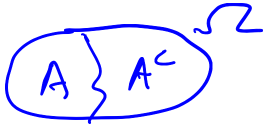
Section 1.4

Consequences of the rules of probability

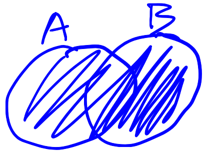
First, some set theory notation:

Complement: $A^c = \{\text{everything not in A}\} = \{\omega \in \Omega : \omega \notin A\}$.



Ex) $A = \{\text{die roll is even}\}$, $A^c = \{\text{die roll is odd}\}$

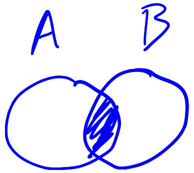
Union: $A \cup B = \{\text{everything in either A or B (or both)}\}$



Ex) $A = \{\text{1st die roll is 1}\}$, $B = \{\text{2nd die roll is 1}\}$

$A \cup B = \{\text{either die roll is 1}\}$

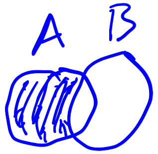
Intersection: $A \cap B = \{\text{everything in both A \& B}\}$



Ex) As above,

$A \cap B = \{\text{both rolls are 1}\}$

Difference: $A \setminus B = \{\text{everything in A but not in B}\} = A \cap B^c$

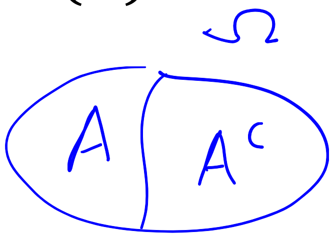


Ex) As above

$A \setminus B = \{\text{1st roll is 1, 2nd is not 1}\}$

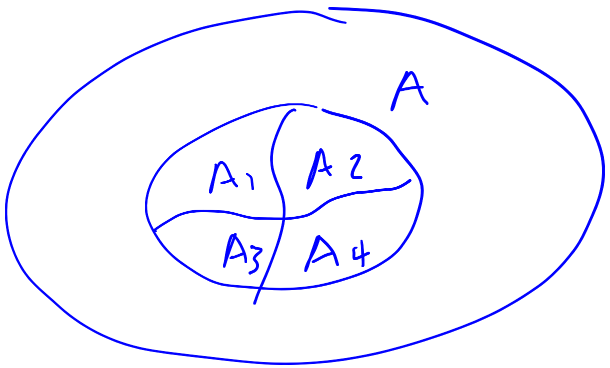
Properties of probability measure

$P(A) = 1 - P(A^c)$.

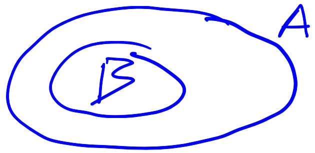


If $A = \cup_{i=1}^{n} A_i$ and A_i are pairwise disjoint, then

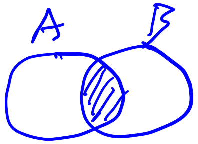
$P(A) = \sum_i P(A_i)$.



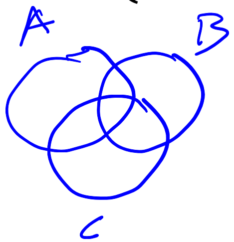
If $B \subset A$, then $P(B) \leq P(A)$.



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$-P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



Inclusion-exclusion principle. See book for version w/ n sets.

Examples

Ex) Roll 3 dice. What is the probability $\underbrace{\text{at least two match}}_{A}$?

$P(A) = 1 - P(A^c) = 1 - P\{\underbrace{\text{none match}}\}$

i.e. all are distinct

$= 1 - \frac{6}{6}\frac{5}{6}\frac{4}{6} = 1 - \frac{20}{36} = \boxed{\frac{16}{36}}$

Alternatively, by inclusion-exclusion.

$B = \{\text{1st \& 2nd dice match}\}$

$C = \{\text{1st \& 3rd match}\}$

$D = \{\text{2nd \& 3rd match}\}$

$A = B \cup C \cup D$

$P(A) = P(B) + P(C) + P(D)$
$- P(B \cap C) - P(B \cap D) - P(C \cap D)$
$+ P(B \cap C \cap D)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} - \frac{1}{36} - \frac{1}{36} - \frac{1}{36} + \frac{1}{36} = \frac{18}{36} - \frac{2}{36} = \boxed{\frac{16}{36}}$

Ex) You have a biased coin that shows heads with probability p and tails with probability $1-p$. Your friend and you take turns flipping, whoever gets heads first wins. Your friend goes first, but you choose p. What value should you choose?

$$\Omega = \{H, TH, TTH, TTTH, \dots\}$$

$$A = \{TH, TTTH, TTTTTH, \dots\} = \{\text{you win}\}$$

$$P(A) = \sum_{\omega \in A} P\{\omega\} = P\{TH\} + P\{TTTH\} + \dots$$

$$= (1-p)\cdot p + (1-p)^3 \cdot p + (1-p)^5 p + \dots$$

$$= (1-p)\cdot p \sum_{k=0}^{\infty} \left[(1-p)^2\right]^k$$

$$= (1-p)\cdot p \cdot \frac{1}{1-(1-p)^2} = \frac{(1-p)\cdot p}{\cancel{1}-\left[\cancel{1}-2p+p^2\right]}$$

$$= \frac{(1-p)p}{2p-p^2} = \frac{1-p}{2-p} \xrightarrow{p \to 0} \frac{1}{2}.$$

$\boxed{\begin{array}{l} \underline{\text{Fact}} \quad \text{summing geometric series. Let } |a|<1. \\ \text{Then} \\ \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} \end{array}}$

Ex) You pick an integer uniformly at random from 1, 2, … , 100. What is the probability that it is $\underbrace{\text{divisible by 10 or by 8}}_{C}$?

$A = \{\text{divisible by } 10\}$

$B = \{\text{divisible by } 8\}$

$C = A \cup B$

$$P(C) = \underbrace{P(A)}_{\frac{10}{100}} + \underbrace{P(B)}_{\frac{12}{100}} - \overbrace{P(\underbrace{A \cap B}_{\text{div. by } 8 \text{ \& } 10 \text{ i.e. } 40})}^{2/100}$$

$$= \frac{10 + 12 - 2}{100} = \frac{20}{100} = \boxed{\frac{1}{5}}$$