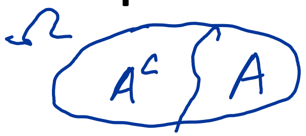
Section 1.4

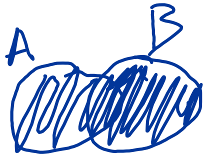
Consequences of the rules of probability

First, some set theory notation:

Complement: $A^c = \{\text{everything not in A}\} = \{\omega \in \Omega : \omega \notin A\}$.



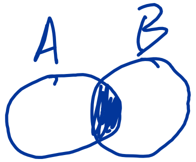
Union: $A \cup B = \{\text{everything in } A \text{ or } B\}$



Ex) Roll two dice, $A = \{\text{1st die} = 1\}$, $B = \{\text{2nd die} = 1\}$

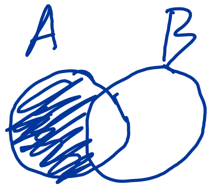
$A \cup B = \{\text{Either die equals } 1\}$

Intersection: $A \cap B = \{\text{everything in } A \text{ \& } B\}$



Ex) As above, $A \cap B = \{\text{Both dice equal } 1\}$

Difference: $A \setminus B = \{\text{everything in } A \text{ but not in } B\} = A \cap B^c$



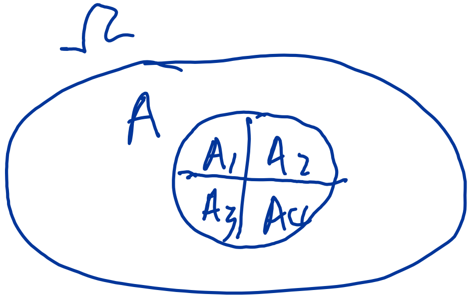
Ex) As above, $A \setminus B = \{\text{1st die is } 1\text{, second is not } 1\}$

Properties of probability measure

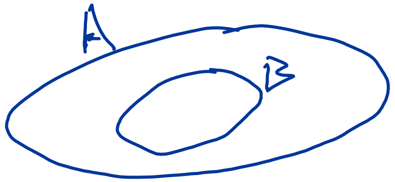
$P(A) = 1 - P(A^c)$.

If $A = \cup_{i=1}^{n} A_i$ and A_i are pairwise disjoint, then

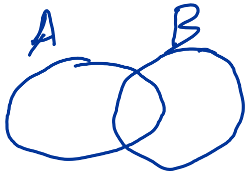
$P(A) = \sum_i P(A_i)$.



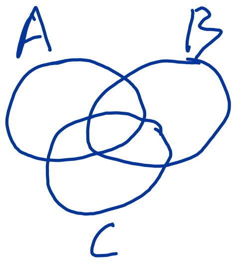
If $B \subset A$, then $P(B) \leq P(A)$.



$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.



$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$



Inclusion-exclusion principle

Examples

Ex) Role 3 dice. What is the probability $\underbrace{\text{at least two match}}_{A}$?

$$P(A) = 1 - P(A^c) = 1 - \underbrace{1 \cdot \frac{5}{6}\,\frac{4}{6}} = \boxed{\frac{16}{36}}$$

like birthday problem

$A^c = \{$Dice have distinct values$\}$

Alternative method

$F = \{$dice 1 & 2 match$\}$

$G = \{$dice 2 & 3 match$\}$

$H = \{$dice 3 & 1 match$\}$

$A = F \cup G \cup H$

By inclusion-exclusion principle

$$P(F \cup G \cup H) = P(F) + P(G) + P(H) - P(F \cap G) - P(G \cap H) - P(F \cap H) + P(F \cap G \cap H)$$

$$= 1/6 + 1/6 + 1/6 - 1/36 - 1/36 - 1/36 + 1/36$$

$$= 1/2 - \frac{2}{36} = \frac{18}{36} - \frac{2}{36} = \boxed{\frac{16}{36}}$$

Ex) You have a biased coin that shows heads with probability p and tails with probability $1-p$. Your friend and you take turns flipping, whoever gets heads first wins. Your friend goes first, but you choose p. What value should you choose?

$$\Omega = \{H, TH, TTH, TTTH, \dots\}$$

$$P(\text{Win}) = P(\{TH, TTTH, TTTTTH, \dots\})$$

$$= P\{TH\} + P\{TTTH\} + P\{TTTTTH\} + \dots$$

$$= (1-p)\cdot p + (1-p)^3 \cdot p + (1-p)^5 p + \dots$$

$$= (1-p)p \underbrace{\left[1 + (1-p)^2 + \left((1-p)^2\right)^2 + \dots\right]}_{\text{geom series}}$$

$$= (1-p)\cdot p \cdot \frac{1}{1-(1-p)^2} = \frac{(1-p)\cdot p}{2p - p^2} = \frac{1-p}{2-p}$$

$$\lim_{p \to 0} P(\text{win}) = \lim_{p \to 0} \frac{1-p}{2-p} = 1/2$$

$a = (1-p)^2$

In brackets:

$$[1 + a + a^2 + \dots] = \frac{1}{1-a}$$

Ex) You pick an integer uniformly at random from $1, 2, \ldots, 100$. What is the probability that it is divisible by 10 or by 8?

$A = \{\text{div. by } 10\}$, $B = \{\text{div by } 8\}$

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{} = \frac{10}{100} + \frac{12}{100} - \frac{2}{100} = \boxed{\frac{20}{100}}$$

incl. excl.

div by 8 & 10 i.e. div by 40