Section 1.3

Infinitely many outcomes

Ex) Flip a coin until tails show. How many flips?

never showing tails

$\Omega = \{1, 2, 3, \ldots\} \cup \{\infty\}$

P{k flips} = P{1st k-1 flips are heads} · P{kth flip is tails}

$= \frac{1}{2^{k-1}} \cdot \frac{1}{2} = \frac{1}{2^k}$

$P\{\infty\} = 1 - P\{1, 2, 3, \ldots\} = 1 - \sum_{k=1}^{\infty} P\{k\} = 1 - \underbrace{\sum_{k=1}^{\infty} \frac{1}{2^k}}_{1} = 0$

Discrete sample spaces

Ω in the previous example was countably infinite. Finite and countably infinite sample spaces are called *discrete*.

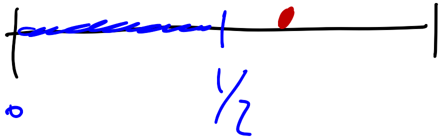
Fact: When the sample space Ω is discrete,
$P(A) = \sum_{\omega \in A} P\{\omega\}$ for any event $A \subset \Omega$.

Proof: $P(A) = P\Big(\underbrace{\bigcup_{\omega \in A} \{\omega\}}_{\text{countable union}}\Big) = \sum_{\omega \in A} P\{\omega\}$

Uncountably infinite sample spaces

Ex) Pick number uniformly at random from $[0,1]$.

- $\Omega = [0,1]$
- $P = ?$



Special cases:

$P([0,1/2]) = \underline{\quad 1/2 \quad}$

$P([0,1/4]) = \underline{\quad 1/4 \quad}$

$P([a,b]) = \text{length}([a,b]) = b-a$ $\quad 0 \le a < b \le 1$

$P([0,\epsilon]) = \epsilon$ for any $\epsilon > 0$

$\Rightarrow P\{0\} \le \epsilon$ for any $\epsilon > 0 \Rightarrow P\{0\} = 0$. Similarly, $P\{\omega\} = 0$ for any $\omega \in [0,1]$.

Can we write $P(A) = \sum_{\omega \in A} P\{\omega\}$?

No: LHS $\neq 0$ in general, RHS $= 0$

Conclusion: In general, for uncountable Ω, $P(A) \neq \sum_{\omega \in A} P\{\omega\}$.

Q: But recall, P must satisfy

$$P(\Omega) = \sum_{i=1}^{\infty} P(A_i)$$

if $A_1, A_2 \ldots$ are pairwise disjoint, and $\cup_i A_i = \Omega$.

Is this a contradiction? No!

$$P(\Omega) = P\Big(\underbrace{\bigcup_{\omega \in \Omega} \{\omega\}}_{\text{disjoint}}\Big) \neq \sum_{\omega \in \Omega} P\{\omega\}$$

when Ω is uncountable