Section 1.3

Infinitely many outcomes

Ex) Flip a coin until tails show. How many flips?

Tails never show

$\Omega = \{1, 2, 3, 4, \ldots\} \cup \{\infty\}$

Tails on 1st flip

Tails on 4th flip

P{k flips} = P{1^{st} k-1 flips are heads} · P{k^{th} flip is tails}

$= \quad \cancel{1/2}\ \frac{1}{2^{k-1}} \quad \cdot \quad 1/2 \quad = \quad \frac{1}{2^k}$

$P\{\infty\} = 1 - P(\{1, 2, \ldots\}) = 1 - \underbrace{\sum_{k \geq 1} \frac{1}{2^k}}_{\text{geometric series}} = 1 - 1 = 0.$

Fact

$$\sum_{k=1}^{\infty} a^k = \frac{a}{1-a} \qquad |a| < 1$$

$$\sum_{k=0}^{\infty} = \frac{1}{1-a}$$

Discrete sample spaces

Ω in the previous example was countably infinite. Finite and countably infinite sample spaces are called *discrete*.

Fact: When the sample space Ω is discrete,
$P(A) = \sum_{\omega \in A} P\{\omega\}$ for any event $A \subset \Omega$.

Proof: $P(A) = P\Big(\underbrace{\bigcup_{\omega \in A} \{\omega\}}_{\text{union of countable number of disjoint sets}}\Big) = \sum_{\omega \in A} P(\omega)$

Uncountably infinite sample spaces

Ex) Pick number uniformly at random from $[0,1]$.

- $\Omega = [0,1]$
- $P = ?$

Special cases:

$P([0,1/2]) =$ $\underline{\quad 1/2 \quad}$

$P([0,1/4]) =$ $\underline{\quad 1/4 \quad}$

$P([0, 1/8]) = 1/8$

$\vdots$

$P([a,b]) = \text{length}([a,b]) = b-a$

$P(\{0\}) \leq P([0,\epsilon]) = \epsilon$ for any $\epsilon > 0 \Rightarrow P(0) = 0$.

Similarly, $P\{w\} = 0$ for any $w \in [0,1]$

Can we write $P(A) = \sum_{\omega \in A} P\{\omega\}$?

RHS: $\sum_{w \in A} P(w) = \sum_{w \in A} 0 = 0$

LHS: Does not $= 0$ in general

Conclusion: In general, for uncountable Ω, $P(A) \neq \sum_{\omega \in A} P\{\omega\}$.

Q: But recall, P must satisfy

$$P(\Omega) = \sum_{i=1}^{\infty} P(A_i)$$

if A_1, A_2 ... are pairwise disjoint, and $\cup_i A_i = \Omega$.

Is this a contradiction? Not a contradiction because

$\sum_{\omega \in A} P\{\omega\}$ is generally an uncountable sum.