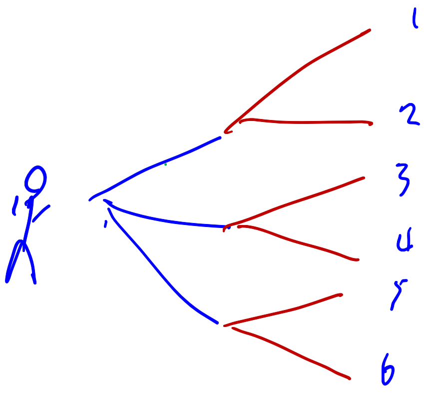
Section 1.2:

Random sampling

Warm up: Counting

You find three paths in the forest. Each of those branches into two paths of their own. How many possible routes might you take?



By enumeration,

$\boxed{ans = 6}$

Alternatively,

$ans = 3 \cdot 2 = 6$

# of 1st paths

# of branches per path

What if there are three paths, each branches into four paths, and each of those branches into five paths? How many total routes?

$ans = 3 \times 4 \times 5 = \boxed{60}$

You role two (six-sided) dice. How many outcomes are there?

<u>6</u> possibilities for 1^{st} die

<u>6</u> possibilities for 2^{nd} die

ans = $6 \cdot 6 = 36$

How many two-letter combinations are there?

<u>26</u> possibilities for 1^{st} letter

<u>26</u> possibilities for 2^{nd} letter

ans = $26 \times 26 = 26^2$

How many two-letter combinations are there where the letters are distinct?

<u>26</u> possibilities for 1^{st} letter

<u>25</u> possibilities for 2^{nd} letter

ans = $26 \cdot 25$

How many two-digit numbers are there with distinct digits? (First digits may be 0.)

$10 \times 9 = 90 = y$

poss. for 1st digit

poss. for 2nd digit

How many two-digit numbers are there so the second digit is larger than the first?

Let x be the answer.

$x \cdot 2 = y = 90$

Since every number w/ ascending digits can be reordered into a number w/ descending digits.

$\Rightarrow \boxed{x = \frac{90}{2} = 45}$

How many ways are there of ordering the numbers 1, 2, 3?

$3 \times 2 \times 1 = 6$

How many ways are there of ordering the numbers 1, 2, 3, …, n?

$n! = n \cdot (n-1)(n-2) \cdots 1$

<u>Fact</u> There are $n!$ ways of ordering a list of n elements

How many three digit numbers are there with distinct digits? (First digit may be 0.)

$10 \cdot 9 \cdot 8 = \underbrace{720 =: y}_{\text{this means set } y \text{ to equal } 720}$

How many three digit numbers are there where the digits are increasing? Let x be answer.

$3!\,x = y$

$\Rightarrow x = \frac{y}{3!} = \frac{720}{6}$

$= \boxed{120}$

Since each number w/ ascending digits can be reordered in 3! ways.

Sampling: Choosing an object at random from a given set

Ex) Deal 5 cards randomly from 52 card deck (sampling). **What is the probability of a full house?**

- Note: Deck has four suits, hearts (H), diamonds (D), clubs (C), and spades (S). There are 13 cards of each suit (some have special names, but let's just call them 1,2,3, …, 13)
- Full house: 3 of one number, and two of another number, e.g.

(5H, 5D, 5S, 9H, 9D)

$$\Omega = \{\underline{\text{ all 5-card poker hands }}\}$$

Two standard choices for defining outcomes:

Option 1: Order matters, so (5H, 5D, 5S, 9H, 9D) is different from
(9H, 9D, 5H, 5D, 5S)

Option 2: Order does not matter (use set notation):

{5H, 5D, 5S, 9H, 9D} = {9H, 9D, 5H, 5D, 5S}

We should be able to answer our question using either option (and get the same answer!)

A B

Note: All outcomes are equally likely (since there is one of each card in deck)

Fact: If $\# \Omega < \infty$ and each outcome is equally likely, then

$P(A) = \dfrac{\# A}{\# \Omega}$

Pf: Step 1. What is $P\{w\}$ for single outcome w?

$$1 = P(\Omega) = P\Big(\underbrace{\bigcup_{w \in \Omega} \{w\}}_{\Omega}\Big) = \sum_{w \in \Omega} P\{w\} = \# \Omega \cdot P\{w\}$$

since $P\{w\}$ is constant.

$\Rightarrow P\{w\} = \frac{1}{\# \Omega}$ for all $w \in \Omega$.

Step 2. $P(A) = P\Big(\bigcup_{w \in A} \{w\}\Big) = \sum_{w \in A} P\{w\} = \# A \cdot P(w) = \# A \cdot \frac{1}{\# \Omega}$ ∎

What is the probability of a full house?

Option 1: Order matters

Counting Ω

Option 1: Order matters

$$\# \Omega = \underline{} 52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 = \frac{52 \cdot 51 \cdots 48 \cdot 47 \cdot 46 \cdots 1}{47 \cdot 46 \cdots 1} = \frac{52!}{47!}$$

Notation: $(n)_k = n(n-1)(n-2)\cdots(n-k+1)$

$\boxed{\text{Fact: There are } (n)_k = \frac{n!}{(n-k)!} \text{ ways of picking } k \text{ distinct elements from a set on } n \text{ when order mattered. Terminology: } \frac{n!}{(n-k)!} \text{ is called "n pick k".}}$

Counting Ω

Option 2: Order doesn't matter

$$\#\Omega = \frac{(52)_5}{5!}$$ because every set of 5 cards can be reordered in 5! ways.

$$= \frac{52!}{47! \cdot 5!}$$

$\boxed{\text{}}$

<u>Fact</u> There are $\frac{n!}{k!(n-k)!}$ ways of choosing k distinct elements out of n, when order doesn't matter. $\binom{n}{k} := \frac{n!}{k!(n-k)!}$ is called "n choose k"

More counting

$A = \{full\ house\},\ \#A = ?$

Option 1: Order matters

$A_{i,j} = \{\text{full houses w/ 3 } i\text{'s and 2 } j\text{'s}\}$

$$A = \bigcup_{i \neq j} A_{i,j}$$

$$\#A = \sum_{i \neq j} \#A_{i,j} = 13 \cdot 12 \cdot \#A_{1,2}$$

since $A_{i,j}$ sets are disjoint

$$\#A_{1,2} = \underbrace{\binom{4}{3}}_{\substack{\text{ways of}\\ \text{choosing 3}\\ \text{suits for}\\ \text{1's}}}\underbrace{\binom{4}{2}}_{\substack{\text{2 suits}\\ \text{for 2's}}}\cdot \overbrace{5!}^{\substack{\text{reorderings}\\ \text{of hand}}}$$

$$A_{1,2} = \{(\underbrace{1\heartsuit, 1\diamondsuit, 1\clubsuit}_{\substack{\text{3 suits out}\\ \text{of 4 for}\\ \text{1's}}}, \underbrace{2\heartsuit, 2\spadesuit}_{\substack{\text{2 suits}\\ \text{out of 4}\\ \text{for 2's}}}), \ldots \}$$

$$\Rightarrow \#A = 13\cdot 12\cdot\binom{4}{3}\binom{4}{2}\cdot 5!$$

$$\Rightarrow \boxed{P(A) = \frac{\#A}{\#\Omega} = \frac{13\cdot 12\cdot\binom{4}{3}\cdot\binom{4}{2}\cdot 5!}{\frac{52!}{47!}}}$$

Question for you: Get same result using Ω where order doesn't matter.