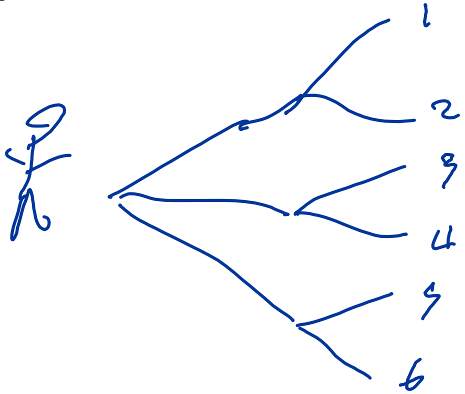
Section 1.2:

Random sampling

Warm up: Counting

You find three paths in the forest. Each of those branches into two paths of their own. How many possible routes might you take?



Ans 6

by enumeration

Alt soln:

$3 \cdot 2 = 6$

poss. for 1st path

poss. for 2nd path

What if there are three paths, each branches into four paths, and each of those branches into five paths? How many total routes?

Ans: $3 \cdot 4 \cdot 5 = 60$

poss for 1st path

poss for 2nd path

poss. for 3rd path

You role two (six-sided) dice. How many outcomes are there?

__6__ possibilities for 1st die

__6__ possibilities for 2nd die

$\# \Omega = 6 \cdot 6 = 36$

How many two-letter combinations are there?

_26_ possibilities for 1st letter

_26_ possibilities for 2nd letter

Soln: $26 \cdot 26 = 26^2$

How many two-letter combinations are there where the letters are distinct?

_26_ possibilities for 1st letter

_25_ possibilities for 2nd letter

Soln: $26 \cdot 25$

How many two-digit numbers are there with distinct digits? (First digits may be 0.)

$10 \cdot 9 = 90 = X$

poss. for 1st digit

poss. for 2nd digit

How many two-digit numbers are there so the second digit is larger than the first?

Call ans to this Y.

Let $\tilde{Y}$ = # of 2-digit numbs w/ second digit smaller.

By symm. $Y = \tilde{Y}$. Further, since every # w/ distinct digits is increasing or decreasing, $Y + \tilde{Y} = X$.

Thus, $X = Y + \tilde{Y} = Y + Y = 2Y$, so $Y = \frac{X}{2} = \frac{90}{2} = \boxed{45}$

How many ways are there of ordering the numbers 1, 2, 3?

$3! = 3 \cdot 2 \cdot 1 = 6$

as below

How many ways are there of ordering the numbers 1, 2, 3, …, n?

$n\,(n-1)\,(n-2)\cdots 1 = n!$

poss for 1st num

poss for 2nd num

$\boxed{\text{Fact: There are } n! \text{ ways of ordering a list of } n \text{ elements.}}$

How many three digit numbers are there with distinct digits? (First digit may be 0.)

$10 \cdot 9 \cdot 8 = X$

1st dig 2nd 3rd

How many three digit numbers are there where the digits are increasing? Call any Y.

$Y \cdot 3! = X \Rightarrow Y = \frac{X}{3!} = \frac{720}{6} = \boxed{120}$

# of orderings of 3-dig number

Sampling: Choosing an object at random from a given set

Ex) Deal 5 cards randomly from 52 card deck (sampling). **What is the probability of a full house?**

- Note: Deck has four suits, hearts (H), diamonds (D), clubs (C), and spades (S). There are 13 cards of each suit (some have special names, but let's just call them 1,2,3, ..., 13)
- Full house: 3 of one number, and two of another number, e.g.

(5H, 5D, 5S, 9H, 9D)

$\Omega = \{$ set of 5-card hands $\}$

Two standard choices for defining outcomes:

Option 1: Order matters, so (5H, 5D, 5S, 9H, 9D) is different from
(9H, 9D, 5H, 5D, 5S)

Option 2: Order does not matter (use set notation):

{5H, 5D, 5S, 9H, 9D} = {9H, 9D, 5H, 5D, 5S}

We should be able to answer our question using either option (and get the same answer!)

Note: All outcomes are equally likely (since there is one of each card in deck)

Fact: If $\# \Omega < \infty$ and each outcome is equally likely, then

$$P(A) = \frac{\# A}{\# \Omega}$$

Pf: $1 = P(\Omega) = P\left(\bigcup_{\omega \in \Omega} \{\omega\}\right) = \sum_{\omega \in \Omega} P(\omega) = \# \Omega \cdot P(\omega)$ for any ω

$\Rightarrow P(\omega) = \frac{1}{\# \Omega}$ for any ω

Then $P(A) = P\left(\bigcup_{\omega \in A} \{\omega\}\right) = \sum_{\omega \in A} P(\omega) = \sum_{\omega \in A} \frac{1}{\# \Omega} = \frac{\# A}{\# \Omega}$ ✓

What is the probability of a full house?

Option 1: Order matters

Counting Ω

Option 1: Order matters

$$\# \Omega = \underbrace{52 \cdot 51 \cdot 50}_{} \, 49 \cdot 48 = \frac{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48 \cdot 47 \cdot 46 \cdots 1}{47 \cdot 46 \cdots 1} = \frac{52!}{47!}$$

poss for ...
1st card

Notation: $(n)_k = n(n-1)(n-2)\cdots(n-k+1) = \frac{n!}{(n-k)!}$

Called "n pick k"

Counting Ω

Option 2: Order doesn't matter

Let x be $\#\Omega$ when order matters,
y be $\#\Omega$ when order doesn't matter.

$y\,5! = x$ i.e. $y = \frac{x}{5!} = \frac{52!}{47!\,5!} = \binom{52}{5}$

Fact The number of ways of choosing k elements from a list of n elements (disregarding their order) is $\frac{n!}{k!(n-k)!} = \binom{n}{k}$. Called "$n$ choose k".

More counting

$A = \{full\ house\},\ \#A = ?$

Option 1: Order matters

Note $A = \bigcup_{\substack{i,j \\ i \neq j}} A_{i,j}$ where $A_{i,j}$ is a full house w/ 3 i's & 2 j's.

$$\#A \overset{\text{since } A_{i,j} \text{ are disjoint}}{=} \sum_{\substack{i,j \\ i \neq j}} \#A_{i,j} \overset{\text{by symm}}{=} \sum_{\substack{i,j \\ i \neq j}} \#A_{1,2} = \#A_{1,2} \cdot 13 \cdot 12$$

since $A_{i,j}$ are disjoint

by symm

$$\#A_{1,2} = \underbrace{\binom{4}{3}}_{\text{suits for 1}} \cdot \underbrace{\binom{4}{2}}_{\text{suits for 2}} \cdot \underbrace{5!}_{\text{orderings w/ suits decided.}}$$

$$A_{1,2} = \{(\underbrace{1H, 1D, 1S}_{\text{3 suits for 1}}, \underbrace{2D, 2S}_{\text{2 suits for 2}}), \ldots\}$$

$$\Rightarrow \boxed{\#A = 13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2} \cdot 5!}$$

$$\boxed{P(A) = \frac{\#A}{\#\Omega} = \frac{13 \cdot 12 \cdot \binom{4}{3} \binom{4}{2} \cdot 5!}{52!/47!}}$$