Section 1.1

Sample spaces and probabilities

Sample space

Definitions illustrated with simple example: Roll two 4-sided dice

<u>Definitions:</u>

Sample point $=$ possible outcome, usually denoted ω.

E.g., $\omega =$ (1, 3)

1st die = 1

2nd die = 3

Sample space $=$ the set of all sample points, denoted Ω

E.g, $\Omega = \{(1,1), (1,2), (1,3), (1,4),$ (2,1) ,...(3,4), (4,4)}

$\#\Omega$ $=$ cardinality of Ω. E.g, $\#\Omega$ = 16

Definitions continued

Event ≔ subset of Ω.

It can often be described with words.

E.g., A = {The dice show the same number} = {(1,1), (2,2), (3,3), (4,4)}

F = {all possible events} = all subsets of Ω

$\#F = 2^{\#\Omega}$

E.g., $F = \{\{\}, \{(1,1)\}, \{(1,1),(1,2)\}, \ldots, \Omega\}$

challenge: prove this

Probability measure: $P: F \rightarrow [0,1]$

- *For event* $A \in F, P(A)$ = *"probability Event A occurs"*
- In our example, $P(\{(1,3)\}) = P\{(1,3)\} = 1/16 = P\{\omega\}$ for any $\omega \in \Omega$, and

we may omit outer bracket

$P\{(1,1),(2,2),(3,3),(4,4)\} = 1/16 + 1/16 + 1/16 + 1/16 = \frac{4}{16} = \frac{1}{4}$

and $P(A) = \frac{\#A}{\#\Omega} = \frac{\#A}{16}$

only holds when each outcome is equally likely!

Definitions continued

$\phi = \{\ \}$

P satisfies:

- $0 \le P(A) \le$ $\underline{1}$, for any $A \in F$
- $P(\emptyset) =$ $\underline{0}$, $P(\Omega) =$ $\underline{1}$
- If $A_1, A_2, A_3, \ldots$ are $\underbrace{\text{pairwise disjoint}}_{A_i \cap A_j = \{\} \quad i \neq j}$, $P(\bigcup_{i=1}^{\infty} A_i) =$ $\underline{\sum_{i=1}^{\infty} P(A_i)}$

A_1 A_2

This implies $P(\bigcup_{i=1}^{n} A_i) =$ $\underline{\sum_{i=1}^{n} P(A_i)}$

disjoint

E.g., $P(\text{First die role equals 1 or 2}) =$ $\underline{P(\{1\text{st die} = 1\} \cup \{1\text{st die} = 2\})}$

$= P\{1\text{st die} = 1\} + P\{1\text{st die} = 2\} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

The triple, (Ω, F, P), is called a *probability space.*

finer point

Q: Flip 3 coins. What is the corresponding probability space?

(Need to define Ω and P)

$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

$P(\text{1st flip is } H) = P\{HHH, HHT, HTH, HTT\} = \frac{4}{8} = 1/2$

heads

For $A \subset \Omega$, $P(A) = \frac{\#A}{8}$

since each outcome is equally likely