The University of British Columbia

Math 302 — Introduction to Probability

Sample midterm

Name: ______________________________ Student ID: ______________

Instructions

- This exam consists of **4 questions** worth a total of 32 points.
- Make sure this exam has **4 pages** excluding this cover page.
- Note that there is a **table of discrete distributions** on Page 1, too.
- **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so). No credit might be given for unsupported answers.
- In the actual midterm, the questions are phrased in such a way that **all answers can be simplified** without the help of a calculator. The questions in this sample midterm are similar to the ones in the actual midterm, but are not optimized in this way, and in a few places, a calculator may be useful.
- No calculators, notes, or other aids will be allowed in the actual midterm.
- If you need more space, use the back of the pages.
- Duration: **50** minutes.

Question	Points	Score
1	6	
2	10	
3	8	
4	8	
Total:	32	

Common Discrete Distributions

Random Variable X	$P(X=k)$	Mean	Variance
Ber(p)	$P(X=0)=1-p$, $P(X=1)=p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
Geom(p)	$p(1-p)^{k-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

6 marks

1. A cutlery set contains 6 knives and 6 forks. 4 pieces of cutlery are chosen at random. Find (do NOT simplify) the probability that:
   (a) 2 complete pairs (of one knife and one fork each) are chosen;
   (b) exactly 1 complete pair (knife + fork) is chosen

Solutions

(a)

$$\frac{\binom{6}{2}^2}{\binom{12}{4}}$$

(b)

$$\frac{2\cdot 6\cdot \binom{6}{3}}{\binom{12}{4}}$$

10 marks 2. 20% of a population are infected by a virus. A medical test correctly identifies all of the infected people, but only 50% of the healthy people.

(a) All people in the population are tested once. What is the percentage of people with positive test?

(b) If an individual is tested positive, what is the probability that he has the virus?

(c) An individual performs a sequence of independent tests, and all their results are positive. After how many positive tests can he be 80% sure that he has the virus.

Solutions

(a) Define the events A = infected and B_i = ith test is positive. Then, by the law of total probability, the percentage is

$$\begin{aligned}\mathbb{P}(B_1) &= \mathbb{P}(B_1|A)\mathbb{P}(A) + \mathbb{P}(B_1|A^c)\mathbb{P}(A^c)\\ &= 1 \cdot 0.2 + 0.5 \cdot 0.8 = 60\%\end{aligned}$$

(b) By Bayes' formula,

$$\begin{aligned}\mathbb{P}(A|B_1) &= \frac{\mathbb{P}(B_1|A)\mathbb{P}(A)}{\mathbb{P}(B_1|A)\mathbb{P}(A) + \mathbb{P}(B_1|A^c)\mathbb{P}(A^c)}\\ &= \frac{0.2}{0.2 + 0.5 \cdot 0.8} = \frac{1}{3}\end{aligned}$$

(c) By Bayes' formula, a person who has been tested positive n times is infected with probability

$$\begin{aligned}\mathbb{P}(A|B_1 \cap \cdots \cap B_n) &= \frac{\mathbb{P}(B_1 \cap \cdots \cap B_n|A)\mathbb{P}(A)}{\mathbb{P}(B_1 \cap \cdots \cap B_n|A)\mathbb{P}(A) + \mathbb{P}(B_1 \cap \cdots \cap B_n|A^c)\mathbb{P}(A^c)}\\ &= \frac{0.2}{0.2 + 0.5^n \cdot 0.8} = \frac{1}{1 + 2^{-n} \cdot 4} = \frac{2^n}{2^n + 4}.\end{aligned}$$

We therefore have to solve

$$\frac{2^n}{2^n + 4} = 0.8 = \frac{16}{20},$$

which gives $n = 4$.

8 marks

3. There are 12 buses in the 100 Mile House bus fleet each with a capacity of 30 people. Currently 6 of the buses are running full, 3 of them have 15 passengers and 3 of them have 5 passengers.

   (a) If a bus is chosen at random what is the probability that the bus is full?

   (b) If a bus rider is chosen at random what is the probability they are on a full bus?

   (c) If a bus rider is chosen at random and X is the number of people on the rider's bus, find the expected value of X.

(a) The number of buses is 12, the number of full buses is 6, they are equally likely, so the probability is $6/12 = 1/2$.

(b) The number of passengers is $6 \cdot 30 + 3 \cdot 15 + 3 \cdot 5 = 240$, 180 of them are on full buses, so the probability is $180/240 = 3/4$.

(c) Similarly to part b) we have

$$\mathbb{P}(X = 30) = \frac{3}{4}, \ \mathbb{P}(X = 15) = \frac{45}{240} = \frac{3}{16}, \text{ and } \mathbb{P}(X = 5) = \frac{15}{240} = \frac{1}{16}.$$

Thus by the definition

$$\mathbb{E}X = 30 \cdot \frac{3}{4} + 15 \cdot \frac{3}{16} + 5 \cdot \frac{1}{16} = \frac{205}{8}.$$

8 marks 4. A child has saved \$10 to buy candy. The kind of candy she likes costs on average \$6 with a variance of $\$^2 2$.

(a) Give a lower bound on the probability that she will be able to buy one piece of candy.

(b) Give a lower bound on the probability that she CANNOT buy three pieces of candy with her budget. Assume that the prices of the three pieces do not depend on each other.

Solutions

(a) Let X_1 be the price of one piece of candy. The event that the child can afford it is $\{X_1 \leq \$10\}$. Since $\{X_1 \leq \$10\} \supset \{|X_1 - 6| \leq 4\}$, we have by Chebyshev

$$\mathbb{P}(X_1 \leq \$10) \geq \mathbb{P}(|X_1 - 6| \leq 4) \geq 1 - \frac{2}{4^2} = \frac{7}{8}$$

(b) Let X_2, X_3 be the prices of the second and third piece of candy. The event that the child cannot afford all pieces is $E = \{X_1 + X_2 + X_3 > 10\}$, and E contains the smaller event

$$E' = \{|X_1 - 6| \leq 2\} \cap \{|X_2 - 6| \leq 2\} \cap \{|X_3 - 6| \leq 3\}.$$

Since the prices are independent, we have

$$\begin{aligned}\mathbb{P}(E) \geq \mathbb{P}(E') &= \mathbb{P}(|X_1 - 6| \leq 2) \cdot \mathbb{P}(|X_2 - 6| \leq 2) \cdot \mathbb{P}(|X_3 - 6| \leq 3) \\ &\geq (1 - \tfrac{2}{2^2}) \cdot (1 - \tfrac{2}{2^2}) \cdot (1 - \tfrac{2}{3^2}) = \frac{7}{36}\end{aligned}$$

Of course, this is a crude lower bound; in reality, we expect a much larger probability. Anything better qualifies for bonus points.