The University of British Columbia

Math 302 — Introduction to Probability

2019, February 13

Name: ______________________________ Student ID: ____________________

Instructions

- This exam consists of **4 questions** worth a total of 32 points.
- Make sure this exam has **4 pages** excluding this cover page.
- Note that there is a **table of discrete distributions** on Page 1, too.
- **Explain** your reasoning thoroughly, and **justify** all answers (even if the question does not specifically say so). No credit might be given for unsupported answers.
- All answers should be **simplified**.
- No calculators, notes, or other aids are allowed.
- If you need more space, use the back of the pages.
- Duration: **50** minutes.

Question	Points	Score
1	7	
2	10	
3	8	
4	7	
Total:	32	

Common Discrete Distributions

Random Variable X	$P(X=k)$	Mean	Variance
Ber(p)	$P(X=0)=1-p,\ P(X=1)=p$	p	$p(1-p)$
Bin(n,p)	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
Geom(p)	$p(1-p)^{k-1}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

7 marks

1. You roll a 6-sided die and flip 6 fair coins. What is the probability that
   (a) You flip 2 Heads and roll a 2.
   (b) The number of Heads flipped is less than the number you rolled. Do NOT simplify.

Solutions:

(a) $\frac{1}{6}\cdot\binom{6}{2}\cdot 2^{-6}$

(b)

$$\tfrac{1}{6}\cdot 2^{-6}\cdot\sum_{k=1}^{6}\sum_{l=0}^{k-1}\binom{6}{l}$$

10 marks 2. You're in love with person A. There is a 25% chance that A is also in love with you. If A is in love with you, A will smile at you every time you meet. If A is not in love with you, A will only smile at you 50% of the time.

(a) A has smiled at you twice. What is the probability that A is in love with you?

(b) A has smiled at you three times. What is the probability that A will smile at you also the next time you meet?

Solutions:

(a) Let F = in love and E_i = smiled at time i. Then

$$\mathbb{P}(E_1 \cap \cdots \cap E_n | F) = 1$$

and

$$\mathbb{P}(E_1 \cap \cdots \cap E_n | F^c) = 2^{-n}$$

and, by the law of total probability

$$\mathbb{P}(E_1 \cap \cdots \cap E_n) = \tfrac{1}{4} + \tfrac{3}{4} \cdot 2^{-n}$$

Thus, by Bayes,

$$\mathbb{P}(F | E_1 \cap E_2) = \mathbb{P}(E_1 \cap E_2 | F) \frac{\mathbb{P}(F)}{\mathbb{P}(E_1 \cap E_2)} = \frac{1}{1 + \frac{3}{4}} = \tfrac{4}{7}.$$

(b) We want

$$\mathbb{P}(E_4 | E_1 \cap \cdots \cap E_3) = \frac{\mathbb{P}(E_1 \cap \cdots \cap E_4)}{\mathbb{P}(E_1 \cap \cdots \cap E_3)} = \frac{1 + 3 \cdot 2^{-4}}{1 + 3 \cdot 2^{-3}} = \frac{19}{22}$$

8 marks 3. Let X be a Geom(p) random variable. Compute $\mathbb{E}g(X)$ where

(a) $g(x) = x \cdot (x-1)$

(b) $g(x) = x \cdot (x-1) \cdot (x-2)$.

Solutions:

(a) We may either compute like

$$\begin{aligned}
\mathbb{E}g(X) &= \sum_{k\geq 1} k(k-1)p(1-p)^{k-1} \\
&= p \sum_{k\geq 0} x \frac{d^2}{dx^2} x^k \bigg|_{x=1-p} \\
&= p \frac{2x}{(1-x)^3}\bigg|_{x=1-p} = 2 \cdot \frac{1-p}{p^2}
\end{aligned}$$

or notice that $\mathbb{E}g(X) = \sigma(X)^2 + \mu^2 - \mu = \frac{1-p}{p^2} + \frac{1}{p^2} - \frac{1}{p}$.

(b) We compute

$$\begin{aligned}
\mathbb{E}g(X) &= \sum_{k\geq 1} k(k-1)(k-2)p(1-p)^{k-1} \\
&= p \sum_{k\geq 0} x^2 \frac{d^3}{dx^3} x^k \bigg|_{x=1-p} \\
&= p \frac{6x^2}{(1-x)^4}\bigg|_{x=1-p} = 6 \cdot \frac{(1-p)^2}{p^3}
\end{aligned}$$

7 marks 4. You are offered to play a game, where you have the opportunity to earn $\$X$, where X is a random variable with $\mathbb{E}X = 100$. The game is risky, so that X can also be negative, and you can afford to lose at most $\$N$.

(a) Suppose that the variance $\sigma(X)^2 = 4000$. How large does N have to be so that you can be 90% sure that you will not lose more than you can afford?

(b) Suppose you can afford to lose at most \$20. In what range does $\sigma(X)^2$ have to lie so that you can be 90% sure this will not happen?

Solutions:

(a) We're looking for the smallest N such that

$$\mathbb{P}(X \le -N) \le 10\%.$$

By Chebyshev,

$$\begin{aligned}\mathbb{P}(X \le -N) &= \mathbb{P}(X - 100 \le -100 - N)\\ &\le \mathbb{P}(|X - 100| \ge 100 + N) \le \frac{4000}{(100+N)^2}\end{aligned}$$

The smallest N which has $\frac{4000}{(100+N)^2} \le 0.1$ is $N = 100$.

(b) We have

$$\mathbb{P}(X \le -20) \le \frac{\sigma(X)^2}{120^2}, \tag{1}$$

and this is smaller than 0.1 if $\sigma(X)^2 \le 1440$.