The University of British Columbia

Math 303: Section 202

2018, February 14

Name: ______________________________ Student ID: ____________________

Instructions

- This exam consists of **4 questions** worth a total of 40 points.
- Make sure this exam has **6 pages** excluding this cover page.
- Explain your reasoning thoroughly, and justify your answers unless the question indicates otherwise.
- No notes, calculators, or other electronic devices are allowed.
- If you need more space, use the back of the pages.
- Duration: **50** minutes.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

10 marks 1. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5, 6\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|ccccccc|} & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ \hline 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{4}{5} & \frac{1}{5} & 0 \\ 4 & 0 & \frac{9}{10} & 0 & 0 & 0 & 0 & \frac{1}{10} \\ 5 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

(a) (2 marks, **no justification needed**) Draw the transition diagram showing the seven states with arrows indicating possible transitions and their probabilities.

(b) (2 marks, **no justification needed**) Determine all the communicating classes of this Markov chain.

Solution: Communicating classes: $\{0, 1, 5\}$, $\{2\}$, $\{3\}$, $\{4\}$, $\{6\}$.

(c) (2 marks, **no justification needed**) Determine which states are recurrent and which are transient.

Solution: Recurrent: $\{0, 1, 5\}$, $\{6\}$; transient: $\{2\}$, $\{3\}$, $\{4\}$.

(d) (2 marks, **no justification needed**) Determine the period of each state.

Solution: Period of $\{0, 1, 5\}$, $\{6\} = 1$, Period of $\{2\}$, $\{3\}$, $\{4\}$ is undefined of ∞.

(e) (2 marks) Suppose the Markov chain started in State 0. What is the probability that it will be in State 0 after 4 steps?

Solution: The possible paths from $0 \to 0$ in 4 steps have probabilities:

$$\mathbb{P}(0 \to 0 \to 1 \to 5 \to 0) = \frac{1}{4},$$
$$\mathbb{P}(0 \to 1 \to 5 \to 0 \to 0) = \frac{1}{4},$$
$$\mathbb{P}(0 \to 0 \to 0 \to 0 \to 0) = \frac{1}{16}.$$

Therefore,

$$P^4_{0,0} = \frac{1}{4} + \frac{1}{4} + \frac{1}{16} = \frac{9}{16}.$$

$\boxed{\text{10 marks}}$ 2. A knight starts at the bottom left of an 8×8 chess board and performs random moves (she can move to a square that is two squares away horizontally and one square vertically, or two squares vertically and one square horizontally). At each state, she picks one of the available legal moves with equal probability, independently of the earlier moves. What is the mean number of moves before she returns to her starting square?

Solution: Consider the connected graph where the squares represent the vertices and two vertices are connected by an edge if there is a legal move between them. For the degrees of this graph it is enough to consider the left bottom 4×4 square by symmetry, see the second picture of `http://www.mathrecreation.com/2011/03/knight-moves.html` for the result. There are 16 vertices of degree 8, 16 of degree 6, 20 of degree 4, 8 of degree 3, and the 4 corners with degree 2. The stationary probabilities of the random walk on graphs were given by Example 4.36, if 0 denotes the left bottom square, we have

$$\pi_0 = \frac{\deg(0)}{\sum_{i=0}^{63} \deg(i)} = \frac{2}{16 \cdot 8 + 16 \cdot 6 + 20 \cdot 4 + 8 \cdot 3 + 4 \cdot 2} = \frac{1}{168}.$$

Thus the mean time of first return is

$$m_0 = \frac{1}{\pi_0} = 168.$$

10 marks 3. Consider a branching process Z_n, $n = 0, 1, \ldots$ with offspring distribution ξ with probability mass function

$$P(\xi = k) = (1-p)^k p, \qquad k = 0, 1, \ldots$$

where $p \in (0, 1)$ is a parameter. Assume $Z_0 = 1$.

(a) (2 marks) What is $\mathbb{E}(Z_n)$?

Solution: Observe that $\xi + 1 \sim \mathrm{Geom}(p)$, so $\mathbb{E}(\xi + 1) = 1/p$ and $\mu = \mathbb{E}(\xi) = 1/p - 1$. Thus

$$\mathbb{E}(Z_n) = \mu^n = \left(\frac{1}{p} - 1\right)^n.$$

(b) (4 marks) What is the probability that the process is extinct by step 3, i.e., what is $\mathbb{P}(Z_3 = 0)$?

Solution: The generating function of ξ is

$$G(s) = \sum_{k=0}^{\infty} (1-p)^k p s^k = p \sum_{k=0}^{\infty} ((1-p)s)^k = \frac{p}{1-(1-p)s}$$

if $|s| < 1/(1-p)$. Therefore,

$$\begin{aligned} P(Z_3 = 0) &= G_3(0) = G(G(G(0))) = G(G(p)) \\ &= G\left(\frac{p}{1-(1-p)p}\right) = \frac{p(1-(1-p)p)}{1-2(1-p)p}. \end{aligned}$$

(c) (4 marks) What is the probability of eventual extinction?

Solution: The probability of eventual extinction η is the smallest nonnegative solution of the equation $G(s) = s$, that is,

$$\frac{p}{1-(1-p)s} = s.$$

Equivalently, we have

$$(p-1)s^2 + s - p = 0.$$

Since $s = 1$ is a root, we can factor out $s - 1$ giving

$$(s-1)((p-1)s + p) = 0,$$

so the nonnegative roots are $s_1 = 1$ and $s_2 = \frac{p}{1-p}$. Thus

$$\eta = \min\left\{1, \frac{p}{1-p}\right\} = \begin{cases} \frac{p}{1-p} & \text{if } p < \frac{1}{2}, \\ 1 & \text{if } p \geq \frac{1}{2}. \end{cases}$$

10 marks 4. Each of two switches is either on or off during a day. On day n, each switch will independently be on with probability

$$\frac{1 + \text{number of switches on during day } (n-1)}{4}.$$

For instance, if both switches are on during day $n-1$, then each will be independently on during day n with probability $3/4$. What fraction of days are both switches on? What fraction are both off?

Solution: Let state i denote number of switches on, then we have a 3-state Markov chain with probability matrix

$$\mathbf{P} = \begin{array}{c|ccc|} & 0 & 1 & 2 \\ \hline 0 & 9/16 & 6/16 & 1/16 \\ 1 & 1/4 & 1/2 & 1/4 \\ 2 & 1/16 & 6/16 & 9/16 \end{array}$$

The fraction of days both switches are off and off is given by the stationary probabilities π_0 and π_0, respectively. We have to solve the system $\pi\mathbf{P} = \pi$ (here we remove one redundant equation, say the last one), and $\sum_i \pi_i = 1$. By Gauss–Jordan elimination:

$$\begin{array}{|ccc|c|} \hline 1 & 1 & 1 & 1 \\ \hline -7/16 & 1/4 & 1/16 & 0 \\ \hline 6/16 & -1/2 & 6/16 & 0 \\ \hline \end{array} \quad \begin{array}{|ccc|c|} \hline 1 & 1 & 1 & 1 \\ \hline -7 & 4 & 1 & 0 \\ \hline 3 & -4 & 3 & 0 \\ \hline \end{array} \quad \begin{array}{|ccc|c|} \hline 1 & 1 & 1 & 1 \\ \hline 0 & 11 & 8 & 7 \\ \hline 0 & -7 & 0 & -3 \\ \hline \end{array}$$

$$\begin{array}{|ccc|c|} \hline 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 3/7 \\ \hline 0 & 0 & 8 & 16/7 \\ \hline \end{array} \quad \begin{array}{|ccc|c|} \hline 1 & 1 & 1 & 1 \\ \hline 0 & 1 & 0 & 3/7 \\ \hline 0 & 0 & 1 & 2/7 \\ \hline \end{array} \quad \begin{array}{|ccc|c|} \hline 1 & 0 & 0 & 2/7 \\ \hline 0 & 1 & 0 & 3/7 \\ \hline 0 & 0 & 1 & 2/7 \\ \hline \end{array}$$

Hence

$$\pi_0 = \frac{2}{7}, \quad \pi_1 = \frac{3}{7}, \quad \pi_2 = \frac{2}{7}.$$