Material covered in class

Week 1: Sections 4.1 and 4.2 from the textbook. Started Subsection 4.5.1.
Keywords: Markov chain (Markov property and stationarity), transition matrix, transition diagram, steady-state (or equilibrium, or invariant, or stationary) distribution, n-step transition probabilities, Chapman-Kolmogorov equations.

Week 2: Gambler's ruin problem from Subsection 4.5.1: we discussed the fair game $p = 1/2$ and the unfair one $p < 1/2$ separately (the case $p > 1/2$ is similar by changing the role of the gambler and the bank). We solved the appearing linear recursion differently from the textbook, using a more general approach. For a brief explanation see `https://brilliant.org/wiki/linear-recurrence-relations` or there are available videos like `https://www.youtube.com/watch?v=aHw7hAAjbD0`.

We solved Exercise 9 (Chapter 4), see also Example 4.12.

Section 4.3 and periodicity. Keywords: state i is accessible from state j, states i and j communicate, communicating classes, irreducible Markov chain, period, periodic and aperiodic states, class property, recurrent and transient states.

Week 3: Section 4.3: Further properties of recurrence and transience.
One-dimensional random walks: recurrent if and only if $p = 1/2$.
Simple random walk in higher dimension: recurrent in dimensions $d = 1, 2$ and transient in dimensions $d \geq 3$. For dimension $d = 2$ our proof slightly differed from the textbook's one, and the textbook does not cover the transience of the SRW in dimensions $d \geq 3$.
Reference: see `http://www.statslab.cam.ac.uk/~james/Markov` for a (partial) lecture notes on Markov chains. Section 1.6 covers a basically similar proof for the recurrence/transience of SRW in dimensions $d = 1, 2, 3$, and the cases $d > 3$ are analogous to the case $d = 3$.

Week 4: Section 4.4: Limiting probabilities, stationary distribution, mean return time, long run proportions and a theorem on their relations. Positive recurrent and null recurrent states, ergodic Markov chain. Examples.

Section 4.8: Reversed Markov chain and its transition probabilities.

Week 5: Section 4.8 continued: Equations for time reversibility, calculating the stationary distribution in case of time reversible chains. Theorem: If the Markov chain is irreducible and finite and π_i satisfies the system $\pi_i P_{i,j} = \pi_j P_{j,i}$ and $\sum_i \pi_i = 1$ then π is the unique stationary distribution. The Ehrenfest urn model: Counting the jumps $i \to j$ and $j \to i$ to obtain time reversibility, guessing and verifying the stationary distribution or solving the time reversible equations. Umbrellas in Exercise 46 (for $r = 3$).

Section 4.7: Branching processes, generating function and its properties, the mean and variance of the size of the nth generation Z_n (denoted by X_n in the textbook).

Week 6: End of Section 4.7: Calculating the extinction probability η as the smallest non-negative root of the equation $s = G(s)$, where G is the generating function of the offspring distribution X. Let $\mu = \mathbb{E}(X)$ and assume that X is not constant 1. Theorem: $\eta = 1$ if $\mu \leq 1$, and $\eta < 1$ if $\mu > 1$. Test 1.

Week 7: Section 5.2: Properties of the exponential distribution: moment generating function, mean, variance, no-memory property, failure rate function, Example 5.3 and similar examples. If $X_1 \sim \text{Exp}(\lambda_1)$ and $X_2 \sim \text{Exp}(\lambda_2)$ are independent then

$$\min(X_1, X_2) \sim \text{Exp}(\lambda_1 + \lambda_2) \quad \text{and} \quad \mathbb{P}(X_1 < X_2) = \frac{\lambda_1}{\lambda_1 + \lambda_2}.$$

Sums of independent random variables, Gamma(n, λ) distribution. First description of Poisson process with rate λ by i.i.d. Exp(λ) random variables.

Week 8: Section 5.3: If $\{N(t)\}_{t \geq 0}$ is a Poisson process with rate λ then for given t we have $N(t) \sim \text{Poisson}(\lambda t)$. The Poisson process has independent and stationary increments. The $o(h)$ notation. A second definition for Poisson process and its equivalence with the first one. The sum of two independent Poisson processes. Examples. Poisson thinning, Proposition 5.2.

Week 9: Subsection 5.3.5: Conditional distribution of arrival times. Order statistics, Theorem 5.2 and its applications: Sampling a Poisson process according to Proposition 5.3. Proof of Proposition 5.3 and Examples 5.18 and 5.20.
Section 6.2: Definition of continuous-time Markov chains (CTMC).

Week 10: Section 6.3: Birth and death processes. Some examples: Poisson process, Yule process, linear growth model with immigration, M/M/1 and M/M/s queuing systems. Example 6.4: For the linear growth model with immigration we calculated the expectation $M(t) = E(X(t))$ by deriving and solving a differential equation for $M(t)$. Section 6.4: Definition of the transition probabilities $P_{ij}(t)$. We proved Kolmogorov's backward equations by using Lemma 6.2 and the Chapman-Kolmogorov equations (Lemma 6.3). Example 6.11: We used Kolmogorov's backward equations to determine the transition probabilities for a CTMC with two states.

Week 11: Kolmogorov's forward equations. Section 6.5: A system of equations for the limiting probabilities by using Kolmogorov's forward equations. Sufficient conditions for the existence. Calculating limiting probabilities for birth and death processes. Midterm 2, holiday.

Week 12: Holiday. Section 6.6: Time reversibility. Embedded Markov chain, relation between π_i and P_i, jump rates and transition probabilities for the backward chain. Time reversibility, detailed balance equations. Proposition 6.5: Birth and death processes are reversible (provided that the limiting probabilities exist). Corollary 6.6 about the output of M/M/s queues.

Week 13: Exercises 30 and 25, long run average of M/M/1 queues ($\lambda < \mu$) and tandem queues. Expectation and distribution of time a customer spends in an M/M/1 queue. Review for the material after the second midterm.