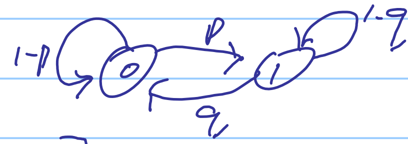
Lecture 9

Note Title 2018-01-21

Examples of time-reversible MC's

Ex) 2-state



$\pi = \left[\frac{q}{q+p}, \frac{p}{q+p} \right]$

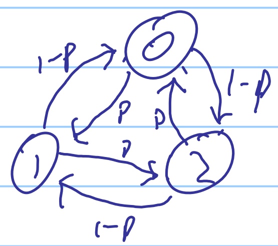
$$\begin{array}{c|cc} & 0 & 1 \\ \hline 0 & 1-p & p \\ 1 & q & 1-q \end{array}$$

Reversible?

$\pi_0 \cdot P_{01} \overset{?}{=} \pi_1 \cdot P_{10}$

LHS $= \frac{q}{q+p} \cdot p$, RHS $= \frac{p}{q+p} \cdot q =$ LHS ✓

Ex)



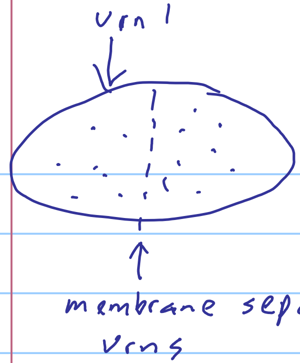
$P = \begin{bmatrix} 0 & p & 1-p \\ 1-p & 0 & p \\ p & 1-p & 0 \end{bmatrix}$

$\pi = \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right]$ (by symmetry)
(unique since it is finite-state irreducible.)

Time reversible?
$\pi_i P_{ij} \overset{?}{=} \pi_j P_{ji}$ $\boxed{\text{only if } p = \tfrac{1}{2}}$

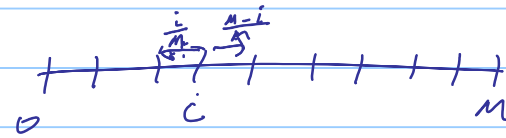
Ex) Ehrenfest chain

M molecules divided between 2 urns.
At each step, select a molecule at random & move it to other urn.
Let $X_n = \#$ of molecules in Urn 1



State space: $\{0, 1, \ldots, M\}$

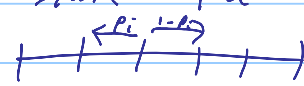
Transition probs:



$P_{i,i+1} = \frac{M-i}{M}$ ← gain molecule

$P_{i,i-1} = \frac{i}{M}$ ← lose molecule

Claim: Any $\overbrace{\text{irreducible finite state}}^{\text{unique stat. dist.}}$ 1-d R.W. w/ arbitrary transition probs



, is time reversible.

$$\underbrace{\underbrace{\pi_i P_{ij}}_{\text{prop jumps from } i \text{ to } j} \overset{?}{=} \underbrace{\pi_j \cdot P_{ji}}_{\text{prop jumps } j \text{ to } i}}_{\text{asymptotically}} \qquad i = j \pm 1$$

But, no $\overline{\text{matter how long}}^{\text{asymptotically}}$ MC has been running, # of transitions from i to j is within 1 of # " " " j to i.

⇒ proportions are asymptotically equal

Goal: Find π, i.e., solve $\pi_i P_{ij} = \pi_j P_{ji}$ $\quad \sum_i \pi_i = 1$

Method 1: Guess and verify.

→ $\text{Bin}(M, \frac{1}{2})$