Lecture 8

Note Title 2018-01-18

- Prove finite state irreducible MC has unique stationary dist.
- Talk about why we care about stationary dist.
- Ask about intuition for time reversal

Time reversal

$$\boxed{\begin{array}{l}\textbf{Thm}\ \text{Given an MC } (X_n)_{0\le n\le N} \text{ w/ stationary dist } \pi \\ \text{\& w/ } P(X_0=j)=\pi_j,\ \text{let } Y_n = X_{N-n}. \\ \text{Then } (Y_n)_{0\le n\le N} \text{ is a MC w/ stationary dist } \pi \\ \text{and transition probs } Q_{ij} = \dfrac{P_{ji}\pi_j}{\pi_i}\end{array}}$$

Pf: Claim 1: Y_n is MC.

We need to show markov property:

$$P(Y_n = i \mid Y_{n-1}=j, Y_{n-2}=k \ldots) \stackrel{?}{=} P(Y_n = i \mid Y_{n-1}=j)$$

$$RHS = P(X_{N-n}=i \mid X_{N-n+1}=j)$$

$$LHS = P(\overset{A}{X_{N-n}=i} \mid \overset{C}{X_{N-n+1}=j}, \overbrace{X_{N-n+2}=k, \ldots}^{B})$$

$$P(A \mid B, C) = \frac{P(B \mid A, C)\cdot P(A \mid C)}{P(B \mid C)}$$

$$= \cancel{\frac{P(X_{N-n+2}=k \ldots \mid X_{N-n}=i, X_{N-n+1}=j)}{P(X_{N-n+2}=k \ldots \mid X_{N-n+1}=j)}} \cdot P(X_{N-n}=i \mid X_{N-n+1}=j)$$

$= RHS.$

Claim 2: Transition probs:

$$Q_{ij} = P(Y_n=j \mid Y_{n-1}=i) = P(X_{N-n}=j \mid X_{N-n+1}=i)$$

$$= P(X_{N-n+1}=i \mid X_{N-n}=j) \cdot \frac{P(X_{N-n}=j)}{P(X_{N-n+1}=i)}$$

$$= P_{ji} \cdot \frac{\pi_j}{\pi_i} \quad \checkmark$$

Since X_n starts in stationary dist. If it didn't we wouldn't have time homogeneity.

Claim 3: π is stationary for Y.

stoch. matrix & rows sum to 1

Indeed $(\pi \cdot Q)_j = \sum_i P_{ji} \frac{\pi_j}{\cancel{\pi_i}} \cdot \cancel{\pi_i} = \sum_i P_{ji} \pi_j = \pi_j \cdot 1 = \pi_j$

$\Rightarrow \pi Q = \pi.$ ■

Def A <u>time reversible</u> MC has $P_{ij} = Q_{ij}$

Prop Let X_n be an irreducible M.C. If $\exists\, x = (x_i)$ s.t. $\sum x_i = 1$ & $x_i P_{ij} = x_j P_{ji}$ then $x = \pi$ is the stationary dist.

Pf: $(x P)_j = \sum_i x_i P_{ij} = \sum_i x_j P_{ji} = x_j \Rightarrow xP = x$.

Also, note $0 \le x_i \le 1 \;\; \forall i$. (why?)