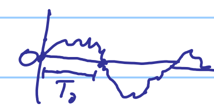
Lecture 7

Note Title 2018-01-16

Limiting probabilities & mean return times

Def Given a recurrent state i, let T_i be the return time to i so, assuming $X_0 = i$, r.v.

$$T_i = \min\{n \geq 1 : X_n = i\}.$$

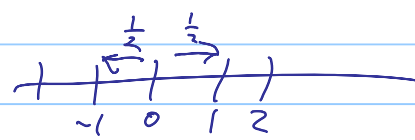


Set $m_i := \mathbb{E}[T_i \mid X_0 = i]$.

We say i is
- positive recurrent if $m_i < \infty$
- null recurrent if $m_i = \infty$.

only possible for ∞ MC.

Ex) 1-d RW

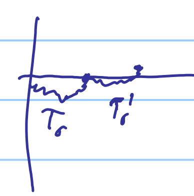


Positive recurrent or null recurrent?
Let T_r be the time it takes to go one step up, e.g., from 0 to 1.

Note: Given $X_0 = 0$, T_0 has the same dist as $T_r + 1$. We will show $\mathbb{E}T_r = \infty$ implying null recurrence. Indeed

$$\mathbb{E}\,T_r = \mathbb{E}[T_r \mid \text{1st step is up}] \cdot P(\text{1st step up}) + \mathbb{E}[T_r \mid \text{1st step is down}] \cdot P(\text{1st step down})$$

$$= 1 \cdot \frac{1}{2} + \mathbb{E}\,2T_r \cdot \frac{1}{2}$$



$$\mathbb{E}T_r = \frac{1}{2} + \mathbb{E}T_r$$

No (finite) soln! $\Rightarrow \mathbb{E}T_r = \infty$.

Prop: positive recurrence & null recurrence are class properties.

Pf: omitted.

Recall: Initial dist: $\alpha_i = P(X_0 = i)$, $\alpha = (\alpha_i)$

Dist after n steps: $\alpha^n := \alpha P^n$

new notation.

α^n_i gives prob $X_n = i$ assuming initial dist α.

Q: What can you say about $\lim_{n\to\infty} \alpha^n$?

Recall: Two state, three cases

not irreducible   not aperiodic $\lim_{n\to\infty} \alpha^n$ exists, is unique

Def An aperiodic positive recurrent MC is called ergodic. A MC is called ergodic if all states are ergodic.

Def A dist π satisfying a) $\pi = \pi P$, b) $\sum \pi_j = 1$, c) $0 \le \pi_j \le 1$ is called a stationary dist.

Note: If the init dist is π, then after n steps the dist is $\pi P^n = \pi P P^{n-1} = \pi P^{n-1} = \dots = \pi$.

Thm For an irreducible, ergodic the stationary MC:

1) a), b) above have a unique soln.

2) $\lim_{n\to\infty} \alpha P^n = \pi$ (it exists, it does not depend on α, it is π).

3) $\pi_j = \frac{1}{m_j}$. In particular, $\pi_j > 0$.

4) $\pi_j = \lim_{n\to\infty} \frac{\#\text{ of visits to } j \text{ by time } n}{n}$

$=$ long run proportion of time spent in state j.

Rmk

If we remove the aperiodicity assumption 1), 3), 4) still hold, but not 2)

Ex) MC w/ trans matrix
$$\begin{array}{c|ccc} & 0 & 1 & 2 \\ \hline 0 & .5 & .4 & .1 \\ 1 & .3 & .4 & .3 \\ 2 & .2 & .3 & .5 \end{array}$$

Find long-run proportion of time in each state.

Soln: solve $\pi = \pi P$, $\pi_0 + \pi_1 + \pi_2 = 1$

$$\Rightarrow \pi = \left(\frac{21}{62}, \frac{23}{62}, \frac{18}{62}\right).$$