2.4 Consequences of concentration

Ex) $f(g) = \|g\|_2$ $\quad g \sim N(0, I_n)$

$L = 1$ by $\triangle \neq$.

$\Rightarrow \| \|g\|_2 - \mathbb{E}\|g\|_2 \|_{\psi_2} \leq c$

$\mathbb{E}\|g\|_2$ is a bit ugly. We can replace it by $\sqrt{\mathbb{E}\|g\|_2^2} = \sqrt{\mathbb{E}\sum_{i=1}^n g_i^2} = \sqrt{n}$ using the following proposition:

Prop Let X be a sub-Gaussian r.v. w/ $\mathbb{E}X > 0$. Then

$$\|X - (\mathbb{E}|X|^p)^{\frac{1}{p}}\|_{\psi_2} \leq c_p \|X - \mathbb{E}X\|_{\psi_2}$$

where $c_p > 0$ only depends on p.

HW Find a "good" bound on c_p and prove the proposition

Return to $f(g) = \|g\|_2$. We have

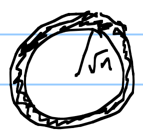
Prop (Concentration of $\|g\|_2$) Let $g \sim N(0, I_n)$. Then $\| \|g\|_2 - \sqrt{n} \|_{\psi_2} \leq C$. Equivalently,

$$P(|\|g\|_2 - \sqrt{n}| > t) \leq \exp(-ct^2) \quad t > 0$$

Take $t = \epsilon\sqrt{n}$ to give:

$$P((1-\epsilon)\sqrt{n} \leq \|g\| \leq (1+\epsilon)\sqrt{n}) \leq \exp(c n \epsilon^2)$$

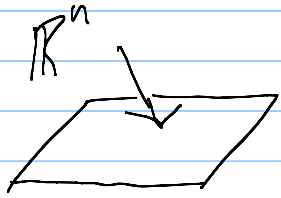
$\Rightarrow$ Gauss vector concentrates near $\sqrt{n}\, S^{n-1}$



HW Sub-Gaussian case. Let $X = (X_1, \dots, X_n)$.
$\mathbb{E} X_i = 0$, $Var(X_i) = 1$, $\|X_i\|_{\psi_2} \le C$.

Does $\|X\|_2$ concentrate around $\sqrt{n}$?
If so, give deviation ineq. If not, find a counter-example.

Ex) Dimension reduction by projection P onto a random k-dimensional subspace



chosen at random uniformly from the Grassmannian $G_{n,k} = \{$all k-dim subspaces in $\mathbb{R}^n\}$

Can be chosen as $span(\vec{X}_1, \dots \vec{X}_k)$
where $X_i \overset{iid}{\sim} Uniform(S^{n-1})$.

Will P preserve the norm of a fixed point $x \in \mathbb{R}^n$.

Prop (Random projection) Let $x \in \mathbb{R}^n$ and P be a random projection onto a k-dim subspace.
Then

(1) $\sqrt{\mathbb{E}\|Px\|_2^2} = \sqrt{\frac{k}{n}}\|x\|_2$

(2) $P\left((1-\epsilon)\sqrt{\frac{k}{n}}\|x\|_2 \le \|Px\|_2 \le (1+\epsilon)\sqrt{\frac{k}{n}}\|x\|_2\right) \ge 1 - 2\exp(-c k \epsilon^2)$

$\epsilon > 0$

Proof: WLOG, $\|x\|_2 = 1 \Rightarrow x \in S^{n-1}$.

$\left\|\begin{matrix}\text{Random projection} \\ \text{of fixed } x\end{matrix}\right\|_2 \overset{dist}{=} \left\|\begin{matrix}\text{Fixed projection} \\ \text{of } x \sim Unif(S^{n-1})\end{matrix}\right\|$

(By rotation invariance)

Thus take $Px = (x_1, x_2, \dots x_k, 0, 0, \dots 0)$

① $\mathbb{E}\|Px\|_2^2 = \sum_{i=1}^{k} \mathbb{E} x_i^2 \overset{*}{=} \frac{k}{n} \sum_{i=1}^{n} \mathbb{E} x_i^2 = \frac{k}{n} \mathbb{E} \sum_{i=1}^{n} x_i^2 = \frac{k}{n}$

*: since x_i are iid

(2) $f(x) = \|Px\|_2$ has Lipschitz const. 1.

HW Complete the proof. (a few lines)

Immediate consequence:

2.5 Johnson-Lindenstrauss Lemma (JL)

Thm Let $\mathcal{X} \subseteq \mathbb{R}^n$ be a set of N points. There exists a linear map $A: \mathbb{R}^n \to \mathbb{R}^k$, $k \le C\epsilon^{-2}\log(N)$, such that

$$(*) \quad (1-\epsilon)\|x-y\|_2 \le \|A(x-y)\|_2 \le (1+\epsilon)\|x-y\|_2 \quad \forall x,y \in \mathcal{X}.$$

Remark: The map $A = \sqrt{\frac{n}{p}}\, P$ satisfies $(*)$ w/ prob. $\geq 1-2\exp(-ck\epsilon^2)$.

Proof: We wish to show that, for each $x \in \mathcal{X}-\mathcal{X} = \{v-w : v,w \in \mathcal{X}\}$,

$$(1-\epsilon)\|x\|_2 \le \|Ax\|_2 \le (1+\epsilon)\|x\|_2 \quad (**)$$

For each fixed x, $(**)$ holds w/ prob $\geq 1-2\exp(-ck\epsilon^2)$ by prop (rand. proj.) above.

By the union bound $(**)$ holds for each $x \in \mathcal{X}-\mathcal{X}$ w/ prob

$$\geq 1 - 2|\mathcal{X}-\mathcal{X}|\exp(-ck\epsilon^2) \geq 1-2N^2\exp(-ck\epsilon^2)$$

$$= 1-2\exp(2\log N - ck\epsilon^2) \geq 1-2\exp(-ck\epsilon^2)$$

by assumption that $k \geq c\epsilon^{-2}\log N$.

Remarks on JL Thm

1) Dimension reduction: From n to $\sim \log|\mathcal{X}|$.

2) Random dim. reduction is linear and non-adaptive

(does not depend on x)

3) Can be used as a lemma to prove RIP etc.

4) The conclusion of JL does not depend on n.

5) Many random linear operators give similar guarantees

HW Prove that $(*)$ in JL thm holds w/ high prob. when $A = \frac{1}{\sqrt{m}} G$, where G is a Gaussian random matrix with iid $N(0,1)$ entries.