Lecture 6

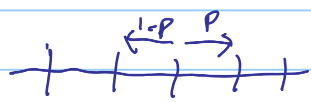
Note Title 2018-01-14

Q: Is random walk on $\mathbb{Z}^d$ transient or recurrent?

Recall:

$$\mathbb{E}N_0 = \sum_{n\geq 0} P_{0,0}^n = \begin{cases} \infty & \text{recurrent} \\ <\infty & \text{transient} \end{cases}$$

$d=1$



$P_{0,0}^n = P^n = P(\text{Get } \frac{n}{2} \text{ heads on } n \text{ biased coin flips})$

↑ abbrev.

$$= \begin{cases} 0 & \text{if } n \text{ odd} \\ \binom{n}{\frac{n}{2}} \cdot p^{n} \cdot (1-p)^{n/2} & \text{if } n \text{ even} \end{cases}$$

Lemma

$$P^{2n} \sim \frac{2^{2n} \cdot p^n \cdot (1-p)^n}{\sqrt{\pi n}}$$

Asymptotic notation: $a_n \sim b_n \iff \lim_{n\to\infty} \frac{a_n}{b_n} = 1$

(a_n, b_n: positive sequences)

Pf of lemma: Stirling approx: $n! \sim \left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}$

$$\Rightarrow \binom{2n}{n} = \frac{2n!}{n!n!} \sim \frac{\left(\frac{2n}{e}\right)^{2n} \cdot \sqrt{2\pi \cdot 2n}}{\left(\left(\frac{n}{e}\right)^n \cdot \sqrt{2\pi n}\right)^2} = \frac{2^{2n}}{\sqrt{\pi n}}.$$

Plug this into $P^{2n} = \binom{2n}{n} \cdot p^n (1-p)^n$ ∎

Thus, $\mathbb{E}N_0 = \sum_{n\geq 0} P^{2n}$ converges iff $\sum_{n\geq 0} \frac{2^{2n}}{\sqrt{\pi n}} (p(1-p))^n =: \sum_{n\geq 0} a_n$ converges.

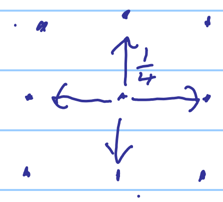
$$a_n = \frac{1}{\sqrt{\pi n}} \cdot \left(4p(1-p)\right)^n$$

Note: if $p \neq \frac{1}{2}$, $4p(1-p) < 1$ & series converges.

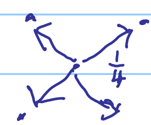
If $p = \frac{1}{2}$, series diverges.

$\Rightarrow$ For $d=1$, R.W. is transient when $p \neq \frac{1}{2}$ and recurrent if $p=\frac{1}{2}$.

<u>$d=2$</u> Assume unbiased jump probs. ($\frac{1}{4}$ chance to jump to each neighbor).



Trick: Rotate by $45°$



$P(\text{up}) = P(\text{down}) = P(\text{right}) = P(\text{left}) = \frac{1}{2}$

$\Rightarrow$ Two simultaneous (indep) 1-d random walks.

$$\Rightarrow P^{2n} = \big(P(\text{1-d random walk returns to origin in } n \text{ steps})\big)^2 \sim \left(\frac{1}{\sqrt{\pi n}}\right)^2 = \frac{1}{\pi n}$$

Since $\sum_{n \geq 0} \frac{1}{\pi n}$ diverges, RW is recurrent.

<u>$d=3$</u> (Move to each neighbor w.p. $\frac{1}{6}$)

$$P^{2n} = \sum_{\substack{j,k \\ j+k \leq 2n}} \frac{2n!}{(j!\,k!\,(n-k-j)!)^2} \left(\frac{1}{6}\right)^{2n}$$

(Summand: take j steps up, j " down, k " right, k " left, $n-k-j$ " forward, " " back)

$$(*) \quad = \underbrace{\left(\frac{1}{2}\right)^{2n} \binom{2n}{n}}_{\sim \frac{1}{\sqrt{\pi n}}} \cdot \sum_{\substack{j,k \\ j+k \leq 2n}} \left(\frac{n!}{j!\,k!\,(n-k-j)!}\right)^2 \cdot \left(\frac{1}{3}\right)^{2n}$$

Note: $$\underbrace{\sum_{\substack{j,k \\ j+k \leq 2n}} \frac{n!}{j!\,k!\,(n-k-j)!} \left(\frac{1}{3}\right)^n}_{\text{sum of probs for } n \text{ rolls of 3-sided die.}} = 1$$ (multinomial expansion)

Thus, $(*)$ is controlled by $$\frac{1}{\sqrt{\pi n}} \cdot \max_{\substack{j,k \\ j+k \leq 2n}} \frac{n!}{j!\,k!\,(n-k-j)!} \cdot \frac{1}{3^n} \quad (**)$$

max is achieved when $j, k, n-k-j$ are as close as.

possible to $\frac{n}{3}$. Then, by stirling approx

$$\frac{n!}{j!\,k!\,(n-k-j)!} \sim \frac{\sqrt{2\pi n}}{\left(\sqrt{2\pi \frac{n}{3}}\right)^3} = \frac{3^{3/2}}{2\pi n}$$

The sequence $\frac{3^{3/2}}{2\pi n} \cdot \frac{1}{\sqrt{\pi n}} = \frac{c}{n^{3/2}}$ ← constant converges!

⇒ for $d=3$, RW is transient.

⇒ for $d \geq 3$ RW is transient since it is transient in 1st 3 dims.