# Lecture 5

Note Title 2018-01-11

## Properties of transience & recurrence

Recall: $f_i = P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i)$
$= P\left(\bigcup_{n \geq 1} A_n \mid X_0 = i\right)$ $\quad A_n = \{X_n = i\}$

$i$ is recurrent if $f_i = 1$
$i$ is transient if $f_i < 1$

$N_i := \#\{n \geq 0 : X_n = i\} =$ "number of returns to $i$" $+1$

**Prop**

If $i$ is recurrent, then
1a) $\mathbb{E}[N_i \mid X_0 = i] = \infty$
2a) $\sum_{n=0}^{\infty} P_{ii}^n = \infty$

3a) if $i \leftrightarrow j$, then $j$ is recurrent.

If $i$ is transient, then
1b) $\mathbb{E}[N_i \mid X_0 = i] < \infty$
2b) $\sum_{n=0}^{\infty} P_{ii}^n < \infty$

3b) if $i \leftrightarrow j$ then $j$ is transient.

4) In a finite state M.C., at least one state is recurrent.

**Pf:**
1a) Recall: $P(N_i = \infty \mid X_0 = i) = 1 \Rightarrow \mathbb{E}[N_i \mid X_0 = i] = \infty$
1b) Recall: $\mathbb{E}[N_i \mid X_0 = i] = \frac{1}{1 - f_i} < \infty$

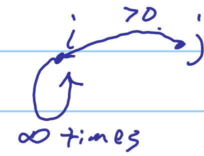
2a), 2b) Let $I_n = \begin{cases} 1 & X_n = i \\ 0 & X_n \neq i \end{cases}$

Then $\mathbb{E}[N_i \mid X_0=i] = \mathbb{E}\left[\sum_{n\geq 0} I_n \mid X_0=i\right] = \sum_{n\geq 0} \mathbb{E}[I_n \mid X_0=i]$

$$= \sum_{n\geq 0} P(X_n=i \mid X_0=i) = \sum_{n\geq 0} P_{ii}^n = \begin{cases} \infty & \text{if recurrent} \\ <\infty & \text{if transient} \end{cases}$$

by 1a), 1b).

3a) We will show that $\sum_{n\geq 0} P_{jj}^n = \infty$




Since $i \leftrightarrow j$ $\exists$ $l, m$ so that $P_{ij}^l > 0$, $P_{ji}^m > 0$.

Then,

$$\sum_{n\geq 0} P_{jj}^n \geq \sum_{n\geq 0} P_{jj}^{n+m+l} = \sum_{n\geq 0} \sum_{s,t} P_{js}^m P_{st}^n \cdot P_{tj}^l$$

(double Chapman Kolmogorov)

$$\geq \sum_{n\geq 0} P_{ji}^m P_{ii}^n P_{ij}^l$$

$$= \underbrace{P_{ij}^m \cdot P_{ji}^l}_{>0} \cdot \underbrace{\sum_{n\geq 0} P_{ii}^n}_{\infty}$$

$$= \infty \quad \blacksquare$$

3a) $\Rightarrow$ 3b)

4) States: $\{1, 2, \dots d\}$. Let $N = \sum_{i=1}^{d} N_i$.

Note $\mathbb{E}[N_i \mid X_0=j] \leq \mathbb{E}[N_i \mid X_0=i]$

$\Rightarrow \infty = \mathbb{E}[N \mid X_0=1] = \sum \mathbb{E}[N_i \mid X_0=1] \leq \sum \mathbb{E}[N_i \mid X_0=i]$

$\Rightarrow \mathbb{E}[N_i \mid X_0=i] = \infty$ for some $i$. $\blacksquare$

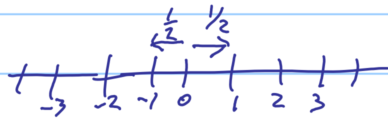
**Def** A M.C. is called <u>irreducible</u> if it has a single communicating class.

**Cor** In a finite-state irreducible M.C., every state is recurrent.

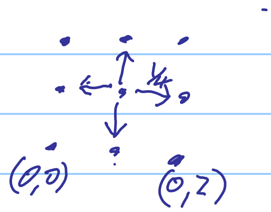
Ex) Simple random walk on $\mathbb{Z}^d$ $\quad d=1,2,3.$

infinite integer lattice

$\mathbb{Z}^1$:




$\mathbb{Z}^2$:




Are the states recurrent?