2. Concentration of measure

2.1 Concentration on the sphere

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$$

σ_n = normalized surface are (probability measure) $\quad \sigma_n(S^{n-1}) = 1$

What is the measure of a equatorial band?

E = equator, say $E = \{x \in S^{n-1} : x_1 = 0\}$

Equatorial band: $E_\epsilon = \{y \in S^{n-1} : \mathrm{dist}(y, E) \leq \epsilon\}$

$\mathrm{dist}(y, E) := \inf_{x \in E} \|y - x\|_2$ $= \{x \in S^{x-1} : |x_1| \leq \epsilon\}$

Low-dimensional case: $n = 2$

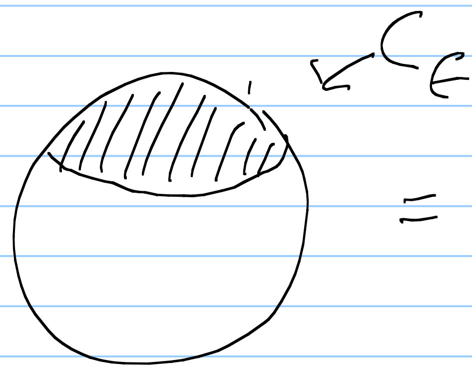
$\Rightarrow \sigma_n(E_\epsilon) = \frac{4 \sin^{-1}(\epsilon)}{2\pi} \approx c \cdot \epsilon$ for $\epsilon \ll 1$

n-dimensional case: (Interesting when $n \gg 1$)

$$\boxed{\text{Prop. } \sigma_n(E_\epsilon) \geq 1 - 2\exp\left(-\frac{n\epsilon^2}{2}\right), \quad \epsilon \geq 0}$$

E_ϵ takes up most of the measure of the sphere if $\epsilon \gtrsim \underbrace{\frac{1}{\sqrt{n}}}_{\text{tiny}}$!

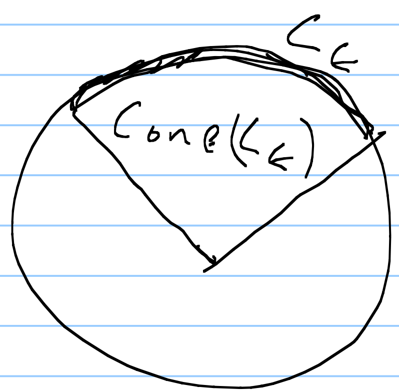
Proof (for small ϵ):

$C_\epsilon :=$ spherical cap =  $= \{x \in S^{n-1} : x_1 > \epsilon\}$

We wish to show that $\sigma_n(C_\epsilon) \le \exp\left(\frac{-n\epsilon^2}{2}\right)$.

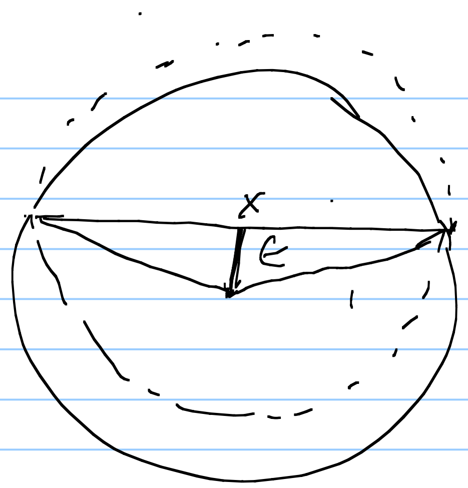
Observe:
$$\sigma_n(C_\epsilon) = \frac{\mathrm{Vol}\left(\mathrm{Cone}(C_\epsilon) \cap B_2^n\right)}{\mathrm{Vol}(B_2^n)}$$

$B_2^n = \ell_2$ unit ball



For $\epsilon \le \frac{1}{2}$, $\mathrm{Cone}(C_\epsilon) \subseteq \overbrace{B(x, \sqrt{1-\epsilon^2})}^{\text{given by dashed line in figure}} = \ell_2$ ball

centered at $x = (\epsilon, 0, \ldots 0)$, w/ radius $\sqrt{1-\epsilon^2}$.



Thus

$$\sigma_n(C_\epsilon) \le \frac{Vol(B(x, \sqrt{1-\epsilon^2}))}{Vol(B_2^n)}$$

$$= \frac{\sqrt{1-\epsilon^2}^{\,n}\, Vol(B_2^n)}{Vol(B_2^n)}$$

$$= (1-\epsilon^2)^{n/2}$$

$$\le \exp\left(-\frac{n\epsilon^2}{2}\right)$$

QED