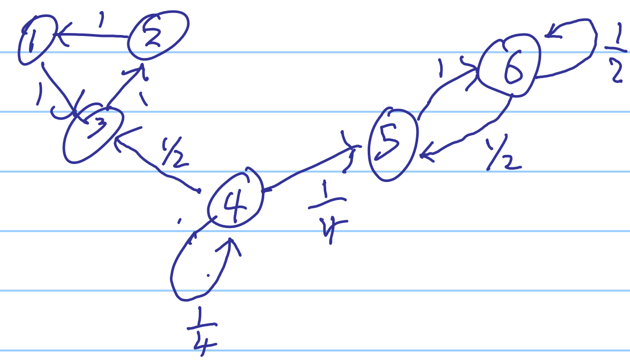
Lecture 4

Note Title 2018-01-09

Ex) transition diagram



Communicating classes:

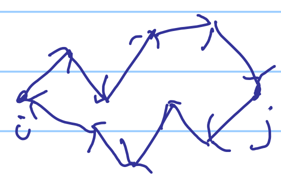
recurrent

$\{1, 2, 3\}, \{4\}, \{5, 6\}$

periodic, period 3 — transient — aperiodic

Def

- j is <u>accessible</u> from i if $P_{ij}^n > 0$ for some $n \geq 0$
- i, j <u>communicate</u>, written $i \leftrightarrow j$, if j is accessible from i & i is accessible from j



Prop Communication is an equivalence relation.

- Reflexive: $i \leftrightarrow i$ since $P_{ii}^0 = 1$.
- Symmetric: If $i \leftrightarrow j$ then $j \leftrightarrow i$ by def.
- Transitive: If $i \leftrightarrow j$ & $j \leftrightarrow k$ then $i \leftrightarrow k$.

Pf: <u>sketch</u>

<u>Technical</u>

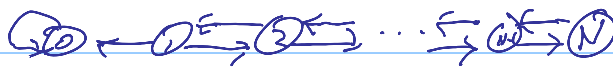
$P_{ij}^{n_1} > 0$ $\quad P_{jk}^{n_2} > 0$ for some n_1, n_2.

$$P_{ik}^{n_1+n_2} = \sum_t P_{it}^{n_1} \cdot P_{tk}^{n_2} \quad \text{(Chapman-Kolmogorov)}$$

$$\geq P_{ij}^{n_1} \cdot P_{jk}^{n_2} > 0$$

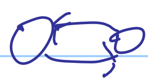
$\Rightarrow$ k is accessible from i. Similarly, i is accessible from k ∎

Thus, the state space always partions into communicating classes!

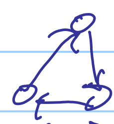
Ex) Gamblers ruin: 

Classes are $\{0\}$, $\{1, 2, \ldots, N-1\}$, $\{N\}$

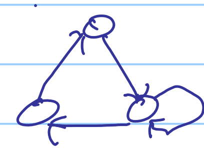
Def i has period d if $P_{ii}^n = 0$ whenever n is not divisible by d, and d is the largest integer w/ this property. I $d=1$, i is called aperiodic.



$d=2$



$d=3$



$d=1$ aperiodic

(Draw arrow where there is nonzero prob)

Prop If i has period d & $i \leftrightarrow j$, then j has period d.

Proof: Exercize for you.

Thus, periodicity is a class property!

Def Let $f_i = P(X_n = i \text{ for some } n \geq 1 \mid X_0 = i)$

i is recurrent if $f_i = 1$.

i is transient if $f_i < 1$

recurrent transient

Prop (Number of return visits)

$$\boxed{\begin{array}{l}\text{Let } N_i = \{\# n \geq 0 : X_n = i\} \quad (\text{random variable}) \\ \text{Then,} \\ \text{① if } i \text{ is recurrent, then } P(N_i = \infty \mid X_0 = i) = 1. \\ \text{② if } i \text{ is transient, } \mathbb{E} N_i = \frac{1}{1-f_i}\end{array}}$$

Pf of ①: $f_i = 1$, so process returns to i. But then it starts fresh and then returns again, etc. ∎

Pf of ②: Similarly, if it returns, it starts fresh. This time $f_i < 1$ so $N_i \sim \text{geom}(1-f_i)$

i.e. $P(N_i = m) = f_i^{m-1}(1-f_i)$.

Thus, $\mathbb{E} N_i = \frac{1}{1-f_i}$. ∎