Lecture 30

Note Title 2018-04-03

CTMC examples

Recall: CTMC is reversible and has limiting probabilities (P_j) if (P_j) satisfy

$$P_i q_{ij} = P_j q_{ji}, \quad P_j \geq 0, \quad \sum_j P_j = 1$$

Q: (Particle on graph) Consider a finite connected undirected (a collection of edges & nodes). On each (i,j) edge there is an indep. Poisson process w/ rate λ_{ij}. If particle is on Node i & event happens on Edge (i,j) particle immediately moves to j (or if on j, moves to i). Its location is CTMC. What are the limiting probs (P_j)?

q_{ij} = rate of going from i to j = λ_{ij}

$q_{ij} = v_i \cdot P_{ij}$ Let $E_i = \{j : (i,j) \text{ is in graph}\}$

v_i: (Time in state i) $= \min_{j \in E_i} \{Exp(\lambda_{ij})\}$

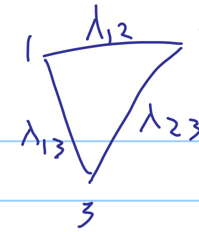
$$\overset{dist}{=} Exp\Big(\underbrace{\sum_{j \in E_i} \lambda_{ij}}_{v_i}\Big)$$

$$P_{ij} = P\Big(Exp(\lambda_{ij}) \leq \min_{\substack{k \in E_i \\ k \neq j}} \{Exp(\lambda_{ij})\}\Big)$$

$$= P\Big(Exp(\lambda_{ij}) \leq Exp\Big(\sum_{\substack{k \in E_i \\ k \neq j}} \lambda_{ij}\Big)\Big) = \frac{\lambda_{ij}}{\sum_{k \in E_i} \lambda_{ij}}$$

$\Rightarrow$ $q_{ij} = v_i \cdot P_{ij} = \lambda_{ij}$.

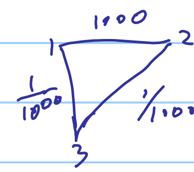
By symmetry, $q_{ji} = \lambda_{ij}$.



Reversibility: $P_i \cancel{q_{ij}} = P_j \cancel{q_{ji}}$ $\quad P_i = P_j = c$

$$\sum_i P_i = 1 \quad \Rightarrow \quad \boxed{P_i = \frac{1}{\#\text{ of nodes}}}$$

Conclusion: Chain is reversible, limiting probs are uniform, spends same proportion of time in each state



Ex) king moves on chess board. It stays in current square for Exp(λ) time. When it moves, it goes

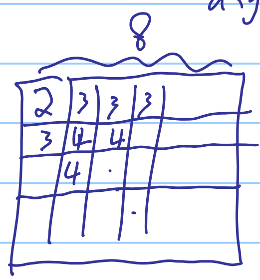
$$\begin{cases} \text{up} & \text{w.p. } \frac{1}{4} \\ \text{down} & \text{w.p. } \frac{1}{4} \\ \text{right} & \text{w.p. } \frac{1}{4} \\ \text{left} & \text{w.p. } \frac{1}{4} \end{cases}$$

. If move is invalid, it stays and waits Exp(λ) time before moving.

Let $P_{(i,j)}$ be the limiting prob for location (i,j) on board.
What is $P_{(1,1)}$?

Soln: Embedded MC is random walk on graph and is thus reversible w/

$\pi_i = c \cdot d_i$
degree of node i



$\pi_{(1,1)} = c \cdot 2$ $\quad$ 4 like this

$\pi_{(1,2)} = c \cdot 3$ $\quad$ 24 like this

$\pi_{(2,2)} = c \cdot 4$ $\quad$ 36 like this

$$P_{(i,j)} = \frac{1}{Z} \cdot \frac{\pi_{(i,j)}}{V(i,j)}$$

$V(2,2) = \lambda$

$V(1,1) = \frac{\lambda}{2}$

$V(1,2) = \frac{3\lambda}{4}$

$$\Rightarrow \quad P_{(1,1)} = \frac{1}{Z} \cdot \frac{C \cdot 2}{\frac{\lambda}{2}} = c' \cdot \frac{4}{\lambda}$$

$$P_{(1,2)} = c' \cdot \frac{3}{\frac{3}{4}\lambda} = c' \cdot \frac{4}{\lambda}$$

$$P_{(2,2)} = c' \cdot \frac{4}{\lambda}$$

i.e. $\boxed{P_{(i,j)} = c'' = \frac{1}{64}}$

Alternate soln: $\quad q_{(i,j),(i',j')} = \frac{\lambda}{4}$. $\quad$ if $(i,j),(i',j')$ are neighbors.