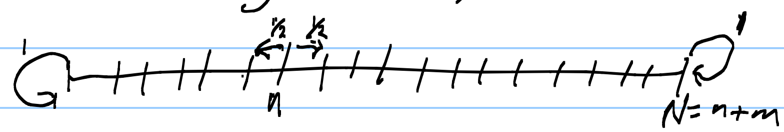
Lecture 3

Note Title 2018-01-07

Example: Gambler's ruin.

Smith has \$n, Bank has \$m. Play 50/50 game betting 1\$ until one goes broke.

Q: P(Smith goes broke) = ?



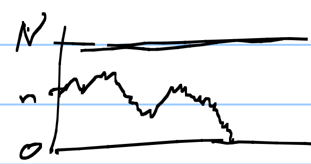
Markov chain: state space $= \{0, 1, \dots N\}$.

state = smith's current fortune

$$P_{i,i-1} = P_{i,i+1} = \tfrac{1}{2} \qquad i = 1, 2, \dots N-1$$

$P_{0,0} = P_{N,N} = 1$ (0 & N are <u>absorbing states</u>)

This is an "simple random walk with absorbing barriers". Alternate motivation: start random walk at n. Find P(hit 0 before N)



<u>Sol'n</u> Let $p(n) = P(\text{Smith goes broke starting at } N)$.

B.C. $p(0) = 1, \quad p(N) = 0$

<u>$1 \le n \le N-1$</u>

$$p(n) = P(\text{broke}) = P(\text{broke} \mid \text{win 1st game}) \cdot P(\text{win 1st game}) + P(\text{broke} \mid \text{lose 1st game}) \cdot P(\text{lose 1st game})$$

$$= p(n+1) \cdot \tfrac{1}{2} + p(n-1) \cdot \tfrac{1}{2}$$

$\Leftrightarrow \; 2p(n) - p(n+1) - p(n-1) = 0$

$\Leftrightarrow \; p(n) - p(n+1) = p(n-1) - p(n)$

True for each $n \Rightarrow$ difference $p(n-1) - p(n)$ is const.

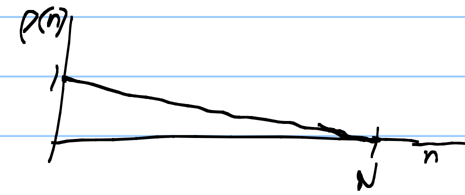
$\Rightarrow p(n) = A + Bn$ for some $A, B \in \mathbb{R}$

Plug in B.C. to find A, B.

$n=0$: $1 = p(0) = A$

$n=N$: $0 = p(N) = 1 + BN \Rightarrow B = -\frac{1}{N}$

Thus, $\boxed{p(n) = 1 - \frac{n}{N}}$



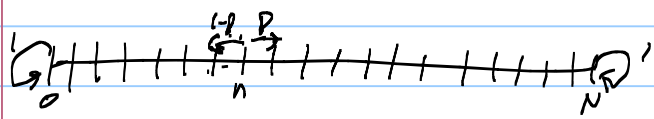
E.g., for $n = 100$, $N = 1000$

$P(\text{smith goes broke}) = 1 - \frac{100}{1000} = 0.9$.

Too risky for bank.

Now, with an unfair game.

$P(\text{smith wins a game}) = p < \frac{1}{2}$



Random walk w/ drift (drunk in wind)

As before, $p(0) = 1$, $p(N) = 0$, $p(n) = p \cdot p(n+1) + (1-p)\, p(n-1)$

2nd order difference eqn: Try $p(n) = x^n$.

$\Rightarrow x^n = p \cdot x^{n+1} + (1-p) x^{n-1}$

$x = p \cdot x^2 + 1 - p$

Note $x = 1$ is a soln.

$p x^2 - x + 1 - p = 0$

$(x-1)(px + p - 1) = 0$

$\Rightarrow x = 1$ or $x = \frac{1-p}{p} =: a$

General soln: $p(n) = A \cdot 1^n + B a^n = A + B a^n$

Plug in B.C.:

$n=0$: $1 = p(0) = A + B$

$n=N$: $0 = p(N) = A + B \cdot a^N$

$\Rightarrow B = \frac{1}{1-a^N}, \quad A = \frac{-a^N}{1-a^N}$

Thus, $p(n) = A + B \cdot a^n = \frac{a^N - a^n}{a^N - 1} = 1 - \frac{a^n - 1}{a^N - 1}$

(Check lim as $p \to \frac{1}{2}$ i.e. $a \to 1$.

$\lim_{a \to 1} p(n) = 1 - \lim_{a\to 1} \frac{a^n - 1}{a^N - 1} = 1 - \underbrace{\lim_{a \to 1} \frac{n \cdot a^{n-1}}{N \cdot a^{N-1}}}_{\text{L'Hôpital}} = 1 - \frac{n}{N}$ recovering $p = \frac{1}{2}$ case.)

Roulette: 18 red, 18 black, 2 green

Smith always bets red so $p = \frac{18}{38} = \frac{9}{19} \approx 0.4737$

$\Rightarrow a = \frac{1 - 9/19}{9/19} = \frac{10}{9}$. Suppose $n = 100$, $N = 1000$.

$P(\text{Smith goes broke}) = 1 - \frac{\left(\frac{10}{9}\right)^{100} - 1}{\left(\frac{10}{9}\right)^{1000} - 1} = 1 - 6.52 \cdot 10^{-42} \approx 1$

Bank is happier.