# Lecture 29

Note Title 3/27/2018

**Recall:** Let $X$ be CTMC in stat. dist.
$Z$ be reverse time CTMC.

Then • $Z$ leaves state $i$ at rate $v_i$ (same as $X$)

• $Z$ has trans matrix $Q_{ij} = \frac{\pi_j P_{ji}}{P_{ij}}$

It is time reversible if

$$P_{ij} = Q_{ij}$$

$$\Leftrightarrow \pi_i P_{ij} = \pi_j P_{ji}$$

$$\Leftrightarrow p_i v_i P_{ij} = p_j v_j P_{ji} \qquad \text{since } p_i = \frac{1}{Z}\frac{\pi_i}{v_i}$$

$\Rightarrow$

**Def** (Detailed balance eqns)

$$\underbrace{p_i q_{ij}}_{\substack{\text{rate of jumps}\\ \text{from } i \text{ to } j}} = \underbrace{p_j q_{ji}}_{\substack{\text{rate of jumps}\\ \text{from } j \text{ to } i}}$$

As in the case of discrete time MC's, this yields a way to find stat dist.

**Prop** Suppose there is a prob. dist. $w = (w_i)$ satisfying the detailed balance eqns, i.e.

$$w_i q_{ij} = w_j q_{ji}, \qquad \sum_i w_i = 1, \qquad w_i \geq 0.$$

Then the CTMC is reversible and $p_j = w_j$

**Prop** $\boxed{\text{An ergodic birth death process is time reversible.}}$

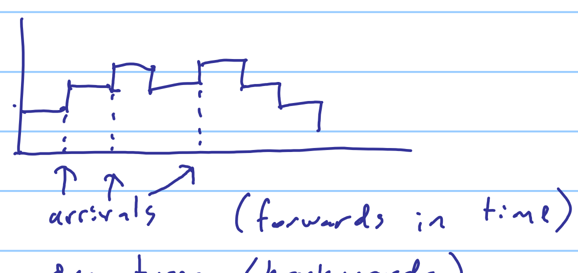
Why? Analog to 1-d random walk.

Q: Consider an M/M/s queue w/ $\lambda < s\mu$.
This is a birth death process w/

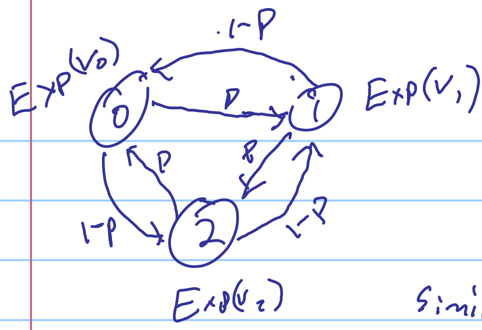
$$\lambda_n \qquad \mu_n = \begin{cases} \mu n & n \le s \\ \mu s & n > s \end{cases}$$

Suppose it is has been running a long time and has achieved equilibrium. Let $\{N(t) : t \geq T\}$ count the number of departures from a certain time $T >> 0$. What is dist of this counting proc?

Soln: Arrivals occur according to rate-$\lambda$ Poisson process. M/M/s queue is reversible, i.e., is the same process going backwards in time (arrivals switch w/ departures). Thus, departures are a rate $\lambda$ Poisson proc.




Q: Reversibility for 3-state MC?




$$w_0 \cdot q_{01} = w_1 \cdot q_{10}$$

$$\iff \boxed{w_0 \cdot p \cdot v_0 = w_1 \cdot (1-p) v_1}$$

similarly $\boxed{w_1 \cdot p \cdot v_1 = w_2 \cdot (1-p) \cdot v_2}$

$$\boxed{w_2 \cdot p \cdot v_2 = w_0 \cdot (1-p) \cdot v_0}$$

Soln?

Take product of the 3 eqns to give

$$(w_1 \cdot w_2 \cdot w_3)(v_1 \cdot v_2 \cdot v_3) \cdot p^3$$
$$= (\quad '' \quad)(\quad '' \quad) \cdot (1-p)^3$$

Soln only if $p = 1-p = \frac{1}{2}$.