Lecture 28

Note Title 3/25/2018

Time reversibility: Let (P_j) be the (unique) limiting probs of CTMC X. Assume X starts in dist (P_j). Let $T >> 0$ and $Z(t) = X(T-t)$ for $0 \le t \le T$.
Q: Rate at which Z leaves state i? Trans probs?
Strategy: Relate to embedded MC. 4 steps.

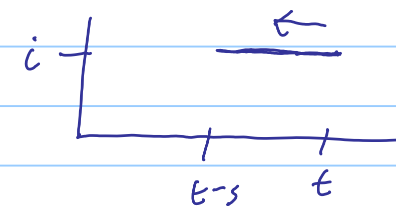
(1) Recall: We set $P'_{ij}(t) = 0$, $P_{ij}(t) = P_j$ in forward Kol. eqns to find eqns for P_j. Thus if CTMC has initial dist (P_j) then after time t,

$$P(X(t)=j) = \sum_i \underbrace{P(X(t)=j \mid X(0)=i)}_{P_{ij}(t)=P_j} \cdot P(X(0)=i)$$

$$= P_j \cdot \cancel{\sum_i P_i}^{\,1} = P_j$$

i.e. (P_j) are stationary probabilities.

(2) Q: Rate that Z leaves state i?



A: Let $0 < s \le t$.

$$P(X(u)=i \ \forall u \in [t-s,t] \mid X(t)=i)$$

$$= \frac{P(X(u)=i \ \forall u \in [t-s,t], \cancel{X(t)=i})}{P(X(t)=i)}$$

$$= \frac{P(X(u)=i \ \forall u \in [t-s,t] \mid X(t-s)=i) \cdot \cancel{P(X(t-s)=i)}^{\,P_i}}{\cancel{P(X(t)=i)}_{\,P_i}}$$

$$= P(\mathrm{Exp}(v_i) > s) = e^{-s v_i}$$

i.e. going backwards time spent in state i is $\text{Exp}(V_i)$, so rate of leaving state i is the same going forwards or backwards.

3) Embedded chain, backwards in time, has trans probs

$$Q_{ij} = \frac{\pi_j P_{ji}}{\pi_i} \qquad (*)$$

Thus, backwards CTMC has trans matrix Q_{ij}.

Subtle point: $(*)$ holds if embedded MC starts is stat. dist π. Further, the reverse time MC is not well defined since it should end for n s.t. $S_n > T$, but this random.

Handwavy soln: Take T very large so dist approaches π an $P(S_n > T) \to 0$.

Technical soln: Define $\{X(t) : -\infty \le t \le \infty\}$, $Z(t) = X(-t)$.