Lecture 27

Embedded MC. Let $\{X(t) : t \geq 0\}$ be CTMC w/ transition probs $\{P_{ij}\}$, rates of leaving states $\{v_i\}$.

Let Y_n be the nth state that $X(t)$ jumps to, i.e. $Y_n = X(S_n)$ where S_n is time of nth jump. This is the embedded MC. It has transition matrix $P = (P_{ij})$.

Q: Suppose $\{Y_n : n \geq 0\}$ has stationary dist $(\pi_0, \pi_1, \ldots)$

Suppose $\{X(t) : t \geq 0\}$ has limiting probs $(P_0, P_1, \ldots)$

$\boxed{\text{Recall: } P_j = \lim_{t \to \infty} P_{ij}(t).}$

What is relationship between π_i & P_i?

Observe: π_i = proportion of visits to i (for X & Y)

$\frac{1}{v_i}$ = avg length of time in state i per visit (for X)

Guess P_i is proportional to $\frac{\pi_i}{v_i}$ i.e.

$$P_i = \frac{1}{Z} \frac{\pi_i}{v_i} \quad \text{w/} \quad Z = \sum_i \frac{\pi_i}{v_i} \qquad (*)$$

This can be verified by checking eqn for P_i:

$$\underbrace{v_j P_j}_{\text{rate leaving } i} = \sum_{i \neq j} P_i q_{ij}$$

Plug (*) into LHS giving $\frac{1}{Z} v_j \cdot \frac{\pi_j}{v_j} = \frac{1}{Z}\pi_j$

Plug (*) into RHS giving $\frac{1}{Z} \sum_{i \neq j} \frac{\pi_i}{\cancel{v_i}} \cancel{v_i} P_{ij} = \frac{1}{Z}(\pi P)_j = \frac{1}{Z}\pi_j$ ✓

(since $\pi P = \pi$)

Conclusion: If (π_i) is stat. dist for Y, then $\left(P_i = \frac{1}{Z_i}\frac{\pi_i}{v_i}\right)$ are limiting probs for X & vice versa.