# Lecture 26

Note Title 2018-03-15

**Recall** $P_k = \lim_{t\to\infty} P_{ik}(t)$ (well defined for irreducible, pos. recurrent CTMC)

Today: Calculate $\{P_k\}$ for birth-death processes

Q: What are $P_k$ for rate $\lambda$ Poisson process $X$?
A: $X(t) \to \infty$ as $t \to \infty$. Thus $P_k = \lim_{t\to\infty} P_{ik}(t) = 0$.
$\Rightarrow \sum_k P_k = 0$.

Why? Null recurrent (and not irreducible).

Q: What are $P_k$ for birth-death process w/ non-zero death rates $\mu_1, \mu_2, \ldots$? ($\mu_0 = 0$)

**Recall:**

$v_j = \lambda_j + \mu_j$

Rate leave $j$ = Rate enter $j$

$$q_{kj} = \begin{cases} \lambda_k & j = k+1 \\ \mu_k & j = k-1 \\ 0 & \text{else} \end{cases}$$

$$v_j \cdot P_j = \sum_{k\neq j} q_{kj} \cdot P_k$$

$$= q_{j-1,j} P_{j-1} + q_{j+1,j} P_{j+1}$$

$$= \lambda_{j-1} P_{j-1} + \mu_{j+1} \cdot P_{j+1}$$

$j=0$ : $\lambda_0 \cdot P_0 = \mu_1 \cdot P_1$

$j=1$ : $(\mu_1 + \lambda_1) P_1 = \lambda_0 P_0 + \mu_2 P_2$

$j=2$ : $(\mu_2 + \lambda_2) P_2 = \lambda_1 P_1 + \mu_3 P_3$

$\vdots$

$j=n$ : $(\mu_n + \lambda_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}$

Solve: $j=0$: $P_1 = \frac{\lambda_0}{\mu_1} P_0$

Plug $j=0$ into $j=1$ $\Rightarrow$ $\lambda_1 P_1 = \mu_2 P_2$

$$\Rightarrow P_2 = \frac{\lambda_1}{\mu_2} \cdot P_1 = \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} \cdot P_0$$

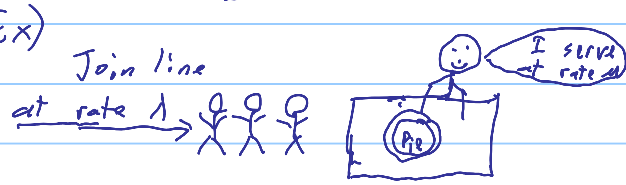
Similarly, $P_3 = \frac{\lambda_2}{\mu_3} \cdot \frac{\lambda_1}{\mu_2} \cdot \frac{\lambda_0}{\mu_1} \cdot P_0$

and $P_n = r_n P_0$ where $r_n = \frac{\lambda_{n-1}}{\mu_n} \cdot \frac{\lambda_{n-2}}{\mu_{n-1}} \cdots \frac{\lambda_0}{\mu_1}$

Now, use $1 = \sum_n P_n = P_0 + P_0 \sum_{n \geq 1} r_n$

$$\Rightarrow \boxed{P_0 = \frac{1}{1 + \sum_{n\geq 1} r_n}, \quad P_n = r_n \cdot P_0 = \frac{r_n}{1 + \sum_{n \geq 1} r_n}}$$

Ex)




Consider a single server who receives customers as a rate $\lambda$ Poisson process. Service times are $\mathrm{Exp}(\mu)$
$\{X(t) : t \geq 0\}$ = # of customers in queue at time $t$ (in service & waiting)

This is called an $M/M/1$ queue.

Markov arrivals — Markov service — # of servers

It is a birth-death process w/ $\lambda_n = \lambda$, $n \geq 0$, $\mu_n = \mu$, $n \geq 1$ $(\mu_0 = 0)$.

In machinery above, $r_n = \frac{\lambda^n}{\mu^n}$ so

Case 1: $\mu > \lambda$ so $\frac{\lambda}{\mu} < 1$

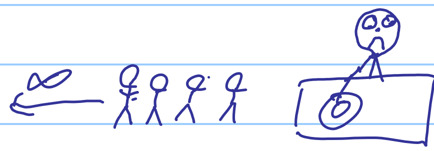
Then, $1 + \sum_{n \geq 1} r_n = 1 + \sum_{n \geq 1} \left(\frac{\lambda}{\mu}\right)^n = \sum_{n \geq 0} \left(\frac{\lambda}{\mu}\right)^n = \frac{1}{1 - \frac{\lambda}{\mu}}$

Thus, $P_0 = 1 - \frac{\lambda}{\mu}$ & $P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$

Case 2: $\mu \geq \lambda$. $\sum_{n \geq 1} r_n = \infty \Rightarrow P_n = 0 \quad \forall n.$

Q: Why?

A: $\lambda > \mu$




grows unbounded

$\lambda = \mu$ Null recurrent, just like symmetric 1-d random walk.