Lecture 25

Note Title 2018-03-13

Limiting probabilities for CTMC

Suppose $P_j = \lim_{t\to\infty} P_{ij}(t)$ exists and does not depend on i

Q: $P_j = ?$ (Find eqns $\{P_j\}$ must satisfy.)

First attempt: Use backward Kol. eqn:

$$P'_{ij}(t) = \sum_{k\neq j} q_{ik} P_{kj}(t) - v_i P_{ij}(t)$$

$$\lim_{t\to 0} \begin{cases} (1)\ P_{ij}(t) \longrightarrow P_j(t) & \text{(by assumption)} \\ (2)\ P'_{ij}(t) \longrightarrow 0 & \text{(follows from (1))} \end{cases}$$

Thus, backward k. eqn gives:

$$0 = \sum_{k\neq j} q_{ik} \cdot P_j - V_i P_j \qquad \text{but } \sum_{i\neq k} q_{ik} = \sum_i q_{ik} = v_i$$

$$= v_i P_j - V_i P_j = 0$$

Not helpful!

Attempt 2:

Forward Kolmogorov eqn

$$\boxed{P'_{ij}(t) = \sum_{k\neq j} P_{ik}(t) \cdot q_{kj} - V_j P_{ij}(t)}$$

Pf (tweak pf of backward k. eqn)

Now take limit as $t\to\infty$:

$$0 = \sum_{k \neq j} P_k \cdot q_{kj} - V_j P_j$$

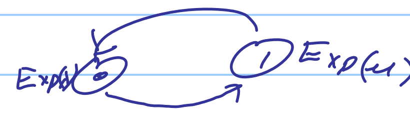
Eqns for $\{P_j\}$

$$\boxed{V_j P_j = \sum_{k \neq j} q_{kj} P_k \, , \qquad \sum_j P_j = 1 \qquad (*)}$$

Ex) Two state MC.

$V_1 = \mu$, $q_{10} = \mu$

$V_0 = \lambda$, $q_{01} = \lambda$



(*) becomes: $V_0 P_0 = q_{10} \cdot P_1$

$$\lambda P_0 = \mu P_1$$

$$P_0 = \frac{\mu}{\lambda} P_1$$

$P_0 + P_1 = 1$

$\frac{\mu}{\lambda} P_1 + P_1 = 1$

$$\boxed{P_1 = \frac{1}{1+\frac{\mu}{\lambda}} = \frac{\lambda}{\lambda+\mu}} \qquad \boxed{P_0 = 1 - P_1 = \frac{\mu}{\lambda+\mu}}$$

Thm For a positive recurrent, irreducible MC

1. the limiting probabilities $\{P_j\}$ exist and obey (*),
2. P_j = long run proportion of time chain is in state j.

Interpretations of (*):

A) Analog to eqn for stationary dist of discrete time MC.

Recall $\pi = \pi P$ $\qquad \sum_i \pi_i = 1$

$\Leftrightarrow \pi_j = \sum_k P_{kj} \pi_j$

B) LHS $= V_j \cdot P_k$ = rate leaving k

RHS $= \sum_{k \neq j} q_{kj} P_j$ = rate coming into k

} must cancel for stationary behaviour