Lecture 24

Note Title 2018-03-11

Recall: Backward Kolmogorov eqn

$$\boxed{\begin{aligned} &P'_{ij}(t) = \sum_{k \neq i} q_{ik} P_{kj}(t) - v_i P_{ij}(t), \quad P_{ij}(0) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \\ &\text{Matrix form} \quad P'(t) = Q\,P(t), \quad P(0) = I \\ &Q = \begin{cases} q_{ij} & i \neq j \\ -v_i & i = j \end{cases} \end{aligned}}$$

Rmk (Soln) $\boxed{P(t) = e^{Qt} = \sum_{n=0}^{\infty} \frac{(Qt)^n}{n!}}$ ← Can be hard to evaluate. Need to diagonalize.

Ex) Birth-death process.

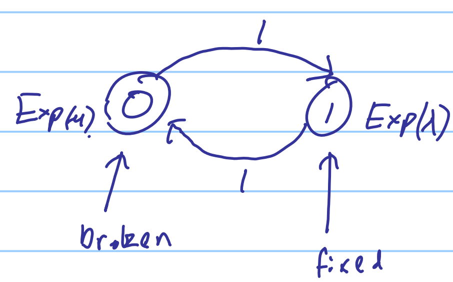
$$P_{i,i+1} = \frac{\lambda_i}{\mu_i + \lambda_i}, \qquad P_{i,i-1} = \frac{\mu_i}{\mu_i + \lambda_i}, \qquad v_i = \mu_i + \lambda_i$$

$$\Rightarrow \quad q_{ij} = v_i P_{ij} = \begin{cases} \lambda_i & j = i+1 \\ \mu_i & j = i-1 \\ 0 & \text{else} \end{cases}$$

So backward eqn is:

$$P'_{ij}(t) = \sum_{k \neq i} \underbrace{q_{ik}}_{\text{need } k = i \pm 1} P_{kj}(t) - v_i P_{ij}(t) = \lambda_i \cdot P_{i+1,j}(t) + \mu_i P_{i-1,j}(t) - (\mu_i + \lambda_i) P_{ij}(t)$$

Special case: 2-state CTMC



Assume: $\begin{cases} \text{time to broken is } Exp(\mu) \\ \text{'' '' fix is } Exp(\lambda) \end{cases}$

Goal: Find $P_{ij}(t)$

This is birth-death process w/

$$\begin{cases} \lambda_0 = \lambda, \quad \mu_1 = \mu \quad \text{otherwise} \quad \mu_i, \lambda_i = 0 \\ P_{10} = 1, \quad P_{01} = 1 \end{cases}$$

so $q_{01} = \lambda, \quad q_{10} = \mu$

$v_0 = \lambda, \quad v_1 = \mu$

Backward eqn: (1) $P'_{00}(t) = \lambda P_{10}(t) - \lambda P_{00}(t)$ $\quad \times \mu$

$P'_{10}(t) = \mu P_{00}(t) - \mu P_{10}(t)$ $\quad \times \lambda$

add

$\longrightarrow \quad \mu P'_{00}(t) + \lambda P'_{10}(t) = 0$

so $\quad \mu P_{00}(t) + \lambda P_{10}(t) = c$

Plug in $t=0$:

$$\mu \overset{1}{\cancel{P_{00}(0)}} + \lambda \overset{0}{\cancel{P_{10}(0)}} = c$$

$\Rightarrow \quad c = \mu$

so $\quad \lambda P_{10}(t) = \mu \left(1 - P_{00}(t)\right)$

Plug into (1): $\quad P'_{00}(t) = \mu\left(1 - P_{00}(t)\right) - \lambda P_{00}(t)$

$$= \mu - (\mu + \lambda) P_{00}(t)$$

Set $f(t) = P_{00}(t)$.

As before: (a) Solve homogeneous part

$$f'(t) = -(\mu+\lambda) f(t) \quad \rightarrow \quad f(t) = C e^{-(\mu+\lambda)t}$$

(b) Particular sol^n: $\quad f(t) = \frac{\mu}{\mu+\lambda}$

Linear combo: $\quad f(t) = C e^{-(\mu+\lambda)t} + \frac{\mu}{\mu+\lambda}$

Init cond: $\quad 1 = f(0) = C + \frac{\mu}{\lambda+\mu} \quad \Rightarrow C = \frac{\lambda}{\lambda+\mu}$

so $\quad \boxed{P_{00}(t) = \frac{\lambda}{\lambda+\mu} e^{-(\mu+\lambda)t} + \frac{\mu}{\lambda+\mu}}$

$$P_{01}(t) + P_{00}(t) = 1$$

$$\Rightarrow \quad P_{01}(t) = 1 - P_{00}(t) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\mu+\lambda)t}\right)$$

By symmetry (exchange λ & μ),

$$P_{11}(t) = \frac{\mu}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\lambda}{\lambda+\mu}, \qquad P_{10}(t) = \frac{\mu}{\mu+\lambda}\left(1 - e^{-(\mu+\lambda)t}\right)$$

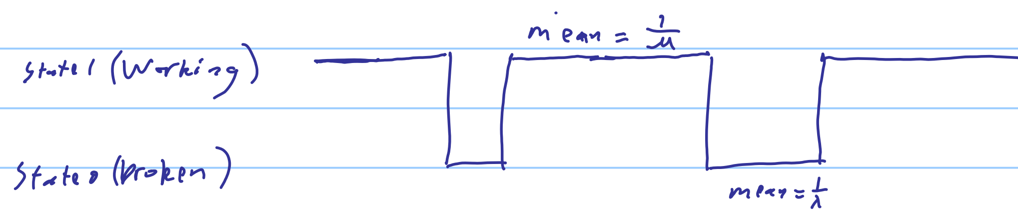
<u>Rmk</u> (Limiting behaviour indep. of initial state.)

Take $t \to \infty$:

$$P_{11}(t),\ P_{01}(t) \longrightarrow \frac{\lambda}{\lambda+\mu}$$

$$P_{00}(t),\ P_{10}(t) \rightarrow \frac{\mu}{\lambda+\mu}$$

Picture:



$$\text{Mean prop of time working} = \frac{\frac{1}{\mu}}{\frac{1}{\mu}+\frac{1}{\lambda}} = \frac{\lambda}{\mu+\lambda} = \lim_{t\to\infty} \begin{cases} P_{01}(t) \\ P_{11}(t) \end{cases}$$