Lecture 23

Note Title 3/8/2018

Q: Consider CTMC $\{X(t): t \geq 0\}$ starting in state i.

1. $P(\geq 2$ jumps by time $h) =$ ____ $+ o(h)$
2. $P(1$ jump by time $h) =$ ____ $+ o(h)$
3. $P(0$ jumps by time $h) =$ ____ $+ o(h)$

For proofs today, assume finite states. Results hold for ∞ states.

Goal: Eqn for transition prob's of CTMC $\{X(t): t \geq 0\}$

$$P_{ij}(t) = P(X(s+t)=j \mid X(s)=i) = P(X(t)=j \mid X(0)=i) = ?$$

Recall: $X(t)$ characterized by:

- v_i = rate of transition from state i (time to leave is $Exp(v_i)$)
- P_{ij} = prob that transition from i goes to j

Write $q_{ij} := v_i \cdot P_{ij}$ = rate of transition from i to j.

Note: $\{q_{ij}\}$ contains all characterizing info, i.e., given $\{q_{ij}\}$, can calculate $\{v_i\}$, $\{P_{ij}\}$. Indeed,

$$\sum_j q_{ij} = v_i \sum_j P_{ij} = v_i, \qquad P_{ij} = \frac{q_{ij}}{v_i}.$$

Lemma (short time transitions)

$$\boxed{\begin{aligned} &(a)\quad P_{ii}(h) = 1 - h v_i + o(h) \\ &(b)\quad P_{ij}(h) = q_{ij} h + o(h). \quad \text{for } i \neq j. \end{aligned}}$$

Pf: (a).

$$P_{ii}(h) = P(\text{no jumps}) + P(\overset{o(h)}{\cancel{\text{jump away and back}}})$$

$$= 1 - h v_i + o(h)$$

(b) $P_{ij}(h) = P(\text{jump}) \cdot P_{ij} = (v_i h + o(h)) \cdot P_{ij} = q_{ij} \cdot h + o(h)$ ✓

Chapman-Kolmogorov eqns:

$$P_{ij}(s+t) = P(X(s+t) = j \mid X(0) = i)$$

$$= \sum_k P(X(s+t) = j \mid X(s) = k, \cancel{X(0)=i}) \cdot P(X(s) = k \mid X(0) = i)$$

$$= \sum_k P(X(t) = j \mid X(0) = k) \cdot P(X(s) = k \mid X(0) = i)$$

$$\boxed{P_{ij}(s+t) = \sum_k P_{ik}(s) \cdot P_{kj}(t)}$$

Kolmogorov backward eqn:

$$P_{ij}(t+h) - P_{ij}(t) = \sum_k P_{ik}(h) \cdot P_{kj}(t) - P_{ij}(t)$$

$$= \sum_{k \neq i} \underbrace{P_{ik}(h)}_{q_{ik} \cdot h + o(h)} \cdot P_{kj}(t) - \underbrace{(1 - P_{ii}(h))}_{v_{ii} h + o(h)} \cdot P_{ij}(t)$$

$$\frac{P_{ij}(t+h) - P_{ij}(t)}{h} = \sum_{k \neq i} q_{ik} + \frac{o(h)}{h} - v_{ii} P_{ij}(t) + \frac{o(h)}{h}$$

Let $h \to 0$:

$$\boxed{P_{ij}'(t) = \sum_{k \neq i} q_{ik} - v_{ii} P_{ij}(t)}$$