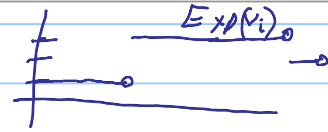
Lecture 22

Note Title 3/6/2018

CTMC: $\{X(t): t \geq 0\}$



Calculate $\mathbb{E}X(t) =: M(t)$ for 2 examples.
Technique: Find differential eqn for $M(t)$ and solve.

Warm up example:

$\{X(t): t \geq 0\}$ is rate λ Poisson process

We will use $M'(t) = \lim_{h \to 0} \frac{M(t+h) - M(t)}{h}$

$$X(t+h) = \begin{cases} X(t)+1 & \text{w.p. } \lambda h + o(h) \\ X(t) & \text{w.p. } 1 - \lambda h + o(h) \\ \text{other} & \text{w.p. } o(h) \end{cases}$$

$$\Rightarrow \quad M(t+h) = \mathbb{E}X(t+h) = (M(t)+1)\lambda h + M(t)(1-\lambda h) + o(h)$$

$$= M(t) + \lambda h + o(h)$$

$$\Rightarrow \quad M'(t) = \lim_{h \to 0} \frac{M(t+h) - M(t)}{h} = \lim_{h \to 0} \frac{\cancel{M(t)} + \lambda h + o(h) - \cancel{M(t)}}{h} = \lambda$$

We have $M'(t) = \lambda$, $M(0) = \mathbb{E}X(0) = 0$

soln $\boxed{M(t) = \lambda t}$ ✓ Recall $M(t) = \mathbb{E}\,\text{Poisson}(\lambda t)$
$= \lambda t$

Ex) Linear Growth Model w/ immigration

$\{X(t): t \geq 0\}$ is a birth & death process w/

$$\left.\begin{array}{l} \cdot\ \mu_n = n \cdot \mu \\ \cdot\ \lambda_n = n\lambda + \theta \end{array}\right\} n \geq 0$$

θ ← immigration rate

Assume population starts w/ i individuals, i.e. $X(0) = i$

Conditional on $X(t)=n$, we have

$$X(t+h) = \begin{cases} n+1 & \text{w.p. } (n\lambda+\theta)h + o(h) \\ n-1 & \text{w.p. } n\mu \cdot h + o(h) \\ n & \text{w.p. } 1-(n\lambda+\theta+n\mu)h+o(h) \\ \text{other} & \text{w.p. } o(h) \end{cases}$$

$$\Rightarrow \mathbb{E}\left[X(t+h) \mid X(t)=n\right] = (n+1)(n\lambda+\theta)h + (n-1)(n\mu h) + n\left(1-(n\lambda+\theta+n\mu)h\right) + o(h)$$
$$= (n\lambda+\theta-n\mu)h + n + o(h) = n + (\lambda-\mu)n \cdot h + \theta \cdot h + o(h)$$

$$\Rightarrow M(t+h) = \mathbb{E}\, X(t+h) = \mathbb{E}\mathbb{E}\left[X(t+h) \mid X(t)\right]$$

$$= \mathbb{E}\, X(t) + (\lambda-\mu)X(t) \cdot h + \theta \cdot h + o(h)$$
$$= M(t) + (\lambda-\mu)M(t) \cdot h + \theta \cdot h + o(h)$$

$$\Rightarrow M'(t) = \lim_{h \to 0} \frac{M(t+h)-M(t)}{h} = \underbrace{(\lambda-\mu)}_{a} \cdot M(t) + \theta =: a M(t) + \theta$$

Initial condition $M(0)=i$

Soln: Case 1 $a \neq 0$

Homogeneous part: $M'(t) = a M(t) \longrightarrow M_H(t) = C e^{at}$

Find a soln to non-homog. part: $M_s(t) = -\frac{\theta}{a}$

Linear combo: $M(t) = C e^{at} - \frac{\theta}{a}$

Plug in init cond: $i = M(0) = C - \frac{\theta}{a} \longrightarrow C = i - \frac{\theta}{a}$

Thus, $M(t) = \left(i - \frac{\theta}{\lambda-\mu}\right) e^{(\lambda-\mu)t} - \frac{\theta}{\lambda-\mu}$

Rmk: If $\lambda > \mu$, exponential growth

$\lambda < \mu$, $\lim_{t \to \infty} M(t) = \frac{\theta}{\mu-\lambda}$

Case 2 $\quad a = 0$ i.e., $r = i$, $\Rightarrow \mu'(t) = 0$

$\Rightarrow \mu(t) = 0t + i$