Lecture 21

Note Title 2018-03-04

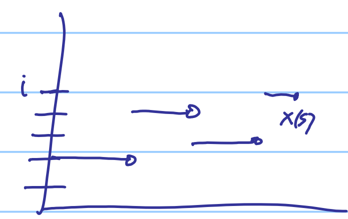
Continuous time Markov Chain (CTMC) : Chapter 6

$\boxed{\text{Def (CTMC) For each } t \geq 0, \text{ let } X(t) \text{ be a r.v. w/ state space } \{0,1,\ldots\}. \text{ Consider the set } \{X(t): t\geq 0\}. \text{ This is a CTMC if it obeys the Markov property, i.e.,}}$

$$\left.\begin{array}{l} P(X(s+t)=j \mid X(s)=i,\ X(u)=x(u) \text{ for } 0\leq s\leq u\} \\ = P(X(s+t)=j \mid X(s)=i) \end{array}\right\} \begin{array}{l}\forall\, s,t\geq 0 \\ i,j\in\{0,1,\ldots\} \\ \{x(u): 0\leq s\leq u\}\end{array}$$

$$= P(X(t)=j \mid X(0)=i)$$

in this class, we assume time homogeneity, i.e. no dependence on s.



We will show that this general def has a surprisingly simple characterization.

Consider interarrival times. Suppose MC is in a given state at given time a. Let T_i be additional time until it leaves i.

Observe: $P(T_i > s+t \mid T_i > s) =$

$$P(X(r)=i \ \forall r\in[a,a+s+t] \mid \cancel{X(r)=i \ \forall r\in[a,a+s]})$$

$X(a+s)=i$ By Markov property

$$= P(T_i > t) \quad \text{by time homogeniity}$$

$\Rightarrow T_i$ is memoryless

Fact: This implies $T_i \sim Exp(\lambda)$ for some λ.

Also, T_i must be indep of which state next jump goes to. Otherwise, this would violate

Markov property, since length of time spent in state would be dependent on next state.

Characterization of CTMC:

For every state $i,j \in \{0,1,\dots\}$

- Time to leave state i: $T_i \sim Exp(V_i)$ for some $V_i > 0$
- Probability of jump from i to j: P_{ij} (Stochastic matrix)

Rmk. Behaves just like (discrete time) MC, except that jumps occur at $Exp(V_i)$ times.

Birth and death processes (Conts version of branching proc)

$X(t)$ = population size at time t

For population of size n

- births occur at rate λ_n
- deaths occur at rate μ_n

Given $\lambda_0, \lambda_1, \dots$ $\mu_0 = 0, \mu_1, \mu_2, \dots$ this defines CTMC w/

$$V_i = \lambda_i + \mu_i$$

$$P_{i,i+1} = \frac{\lambda_i}{\mu_i + \lambda_i}, \quad P_{i,i-1} = \frac{\mu_i}{\mu_i + \lambda_i}$$

Examples:

- Poisson process

$$\mu_n = 0, \quad \lambda_n = \lambda \quad \forall n$$

- Yule process

$$\mu_n = 0, \quad \lambda_n = n \cdot \lambda \quad \forall n$$