Lecture 20

Note Title 2018-03-01

Poisson processes, using what we've learned.

Q: Let $\{N(t): t \geq 0\}$ be a rate λ Poisson process. Let $S_1, S_2, \ldots$ be time of ith event. Find

$$P(S_1, S_2, S_3 \in [0,1] \mid N(10) = 3).$$

A: Given $N(3)=3$, first 3 events occur at uniform at random times in $[0,10]$.
Let $U_1, U_2, U_3 \overset{iid}{\sim} \text{Unif}[0,3]$. We calculate

$$P(U_1, U_2, U_3 \in [0,1]) = P(U_1 \in [0,1])^3 = \left(\tfrac{1}{10}\right)^3 = \tfrac{1}{1000}.$$

<u>Application</u> (HIV analytics)

<u>Model</u>

- individuals contract HIV as a rate λ Poisson process
- time from infection to onset of symptoms is r.v. w/ known cdf G.

Let $\underset{\text{observed}}{N_1(t)}$ = # who have had onset of symptoms
$N_2(t)$ = # who are HIV positive w/out symptoms

Goal: Estimate $\mathbb{E}\, N_2(t)$.

<u>Soln</u>: Consider fixed $t > 0$ and consider fractured Poisson process

Type 1: Infected individual shows symptoms by time t
Type 2: " " doesn't show " " "

Individual infected at time $s \leq t$ is Type 1 w/

prob $P_1(s) = P(\text{onset of symptoms within } t-s \text{ of infection}) = G(t-s)$

Type 2 w/ prob $1 - P_1(s) = 1 - G(t-s) =: \bar{G}(t-s)$

$\Rightarrow N_2(t) \sim \text{Poisson}\left(\lambda \int_0^t P_2(s)\,ds\right)$

Note $\int_0^t P_2(s)\,ds = \int_0^t \bar{G}(t-s)\,ds \underset{\substack{\uparrow \\ u=t-s}}{=} \int_0^t \bar{G}(u)\,du$

$\Rightarrow N_2(t) \sim \text{Poisson}\left(\lambda \underbrace{\int_0^t \bar{G}(u)\,du}_{\text{known}}\right)$

Similarly, $N_1(t) \sim \text{Poisson}\left(\lambda \int_0^t G(u)\,du\right)$

Need to estimate λ: $N_1(t) = n_1$ is observed.

Estimate: $n_1 \approx \mathbb{E} N_1(t) = \lambda \int_0^t G(u)\,du$

Estimate of λ: $\hat{\lambda} = \dfrac{n_1}{\int_0^t G(u)\,du}$

Estimate of $N_2(t)$: $\hat{n}_2 = \hat{\lambda} \int_0^t \bar{G}(u)\,du$

$$= n_1 \cdot \frac{\int_0^t \bar{G}(u)\,du}{\int_0^t G(u)\,du}$$

E.g. $t = 16$ yrs, $G \sim \text{Exp}\left(\frac{1}{10\,\text{yrs}}\right)$ $n_1 = 220{,}000$

$$\hat{n}_2 = 220{,}000 \frac{\int_0^t e^{-u/10}\,du}{\int_0^t (1 - e^{-u/10})\,du} \approx 218{,}959.$$