Chapman-Kolmogorov eqns

Def Given a Markov chain $(X_n)_{n\geq 0}$ the transition matrix P satisfies

$$P_{ij} = P(X_{n+1}=j \mid X_n = i)$$

Rmk Properties of P

① P has as many rows/cols as there are states (possibly ∞).

② $0 \leq P_{ij} \leq 1$, $\sum_j P_{ij} = 1 \quad \forall i$ (rows sum to 1)

matrices that satisfy ② are called stochastic matrices.

Q: ① What is $P(X_{k+n}=j \mid X_k = i)$?

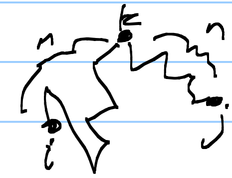
② " " $P(X_n = j \mid X_0 = i)$?

① = ② by stationarity.

Let $P^n_{ij} := P(X_n = j \mid X_0 = i)$.

$n=0$. $P_{ij} = P(X_0 = j \mid X_0 = i) = \delta_{ij} \Rightarrow P^0 = I$

$n=1$. $P^1_{ij} = P_{ij}$ by def.



$$P^{n+m}_{ij} = P(X_{n+m} = j \mid X_0 = i)$$

$$= \sum_k P(X_{n+m} = j, X_m = k \mid X_0 = i)$$

$$= \sum_k P(X_{n+m} = j \mid X_m = k, \cancel{X_0 = i}) \cdot P(X_m = k \mid X_0 = i)$$

Markov prop.

$$= \sum_k P^n_{kj} \cdot P^m_{ik} = \sum_k P^m_{ik} \cdot P^n_{kj}$$

$$\boxed{P^{m+n}_{ij} = \sum_k P^m_{ik} P^n_{kj}}$$ Chapman-Kolmogorov eqn.

E.g., $P^2_{ij} = \sum_k P_{ik} \cdot P_{kj} = (\underbrace{P \cdot P}_{\text{matrix multiplication}})_{ij} = (P^2)_{ij}$

By induction, $P^n_{ij} = (\underbrace{P \cdot P \cdots P}_{n})_{ij} = (P^n)_{ij}$

In other words, the transition probabilities for taking n steps are just P^n.

What if we have random initial state, i.e.,
$P(X_0 = i) =: \alpha_i \qquad \sum_i \alpha_i = 1, \qquad \alpha = (\alpha_1, \alpha_2, \ldots)$

Then, $P(X_n = j) = \sum_i P(X_n = j \mid X_0 = i) \cdot P(X_0 = i)$

$$= \sum_i \alpha_i \cdot P^n_{ij} = (\alpha P^n)_j$$