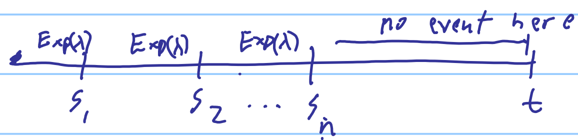
Lecture 19

Note Title 2018-02-27

Conditional dist of arrival times & fractured Poisson processes.

Q: Let $\{N(t): t \geq 0\}$ be a rate λ Poisson process.
Let $S_1, S_2, \dots S_n$ be first n interarrival times.
Note $S_1 \sim Exp(\lambda)$, $S_2 - S_1 \sim Exp(\lambda)$, $\dots S_n - S_{n-1} \sim Exp(\lambda)$ are indep.
What is dist of $S_1, \dots S_n$ given $N(t) = n$.



The conditional joint density

$$f(S_1, S_2 \dots S_n \mid N(t) = n) = ?$$

Intuition (non-rigorous)

$$\frac{P(S_1 \approx s_1, S_2 \approx s_2, \dots S_n \approx s_n, N(t) = n)}{P(N(t) = n)} = \frac{P(S_1 \approx s_1, \overbrace{S_2 - S_1}^{Exp(\lambda)} \approx s_2 - s_1, \dots S_n - S_{n-1} \approx s_n - s_{n-1}, S_{n+1} > t - s_n)}{P(N(t) = n)}$$

$$\approx \frac{\lambda e^{-\lambda s_1} \cdot \lambda e^{-\lambda(s_2 - s_1)} \dots \lambda e^{-\lambda(s_n - s_{n-1})} \cdot e^{-\lambda(t - s_n)}}{\frac{(\lambda t)^n}{n!} \cdot e^{-\lambda t}}$$

$$= \frac{n!}{t^n}$$

A: $f(s_1 \dots s_n \mid N(t) = n) = \frac{n!}{t^n}$ on support $0 \leq s_1 \leq s_2 \leq \dots \leq s_n \leq t$

Note: the density does not depend on $s_1, \dots s_n$
i.e. it is uniform on its support.

Interpretation: The n events, unordered, are conditionally indep. and uniform on $[0, t]$.

To understand this, we need "order statistics".

Def Let $Y_1, \dots Y_n$ be r.v.'s. Their <u>order statistics</u> $Y_{(1)}, Y_{(2)}, \dots Y_{(n)}$ are the values of $Y_1, \dots Y_n$ in increasing order.

Ex) $Y_1 = 2, \quad Y_2 = 1 \quad Y_3 = 3$

$\to \quad Y_{(1)} = 1 \quad Y_{(2)} = 2 \quad Y_{(3)} = 3$

Q: Suppose Y_1, Y_2, Y_3 are iid w/ $P(Y_i = 1) = P(Y_i = 2) = P(Y_i = 3) = \frac{1}{3}$

What is $P(Y_{(1)} = 1, Y_{(2)} = 2, Y_{(3)} = 3)$?

A: $P(Y_1 = 1, Y_2 = 2, Y_3 = 3) \cdot \underset{\substack{\uparrow \\ \text{possible orderings}}}{3!} = \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} 6 = \frac{2}{3}$

Suppose Y_i are iid cont. r.v.'s w/ pdf f.
Then the joint pdf of $Y_{(1)}, Y_{(2)}, \dots Y_{(n)}$ is:

$$f(y_1, \dots y_n) = n! \prod_{i=1}^{n} f(y_i) \qquad y_1 \le y_2 \le \dots \le y_n$$

In particular, if $y_i \sim \text{unif}[0,t]$ then RHS above is

$$\frac{n!}{t^n}.$$

Q: Let $\{N(t) : t \geq 0\}$ be a rate 10^{-10} Poisson process.
What is the prob that the first event occurs in $[0,1]$ given that 3 events occur in $[0,10]$?

A: $P(\min(\underbrace{U_1, U_2, U_3}_{\text{iid unif}[0,10]}) \in [0,1]) = 1 - P(U_1, U_2, U_3 \geq 1) = 1 - \left(\frac{9}{10}\right)^3.$

Now consider a Poisson process in which each event can be of k possible types & for each time s there is a distribution $P_i(s)$ $\quad i = 1, \dots k$ s.t.

$P(\text{event occuring at time } s \text{ is Type } i) = P_i(s)$

(type selection is indep of everything previous)

Prop Let $N_i(t) = \#$ of Type i events by time T.
Then $N_1(t), \dots N_k(t)$ are indep Poisson r.v.'s
w/ means

$$\mathbb{E}\, N_i(t) = \lambda \int_0^t P_i(s)\, ds$$

Pf: $P(N_1(t)=n_1, \dots N_k(t)=n_k) = P(N_1(t)=n_1, \dots N_k(t)=n_k \mid N(t)=n) \cdot P(N(t)=n)$ (*)

$n = n_1 + n_2 + \dots + n_k$

Given $N(t)=n$, there are n indep Unif$[0,t]$ r.v.'s giving event times. Prob that event is type i is $p_i = \frac{1}{t}\int_0^t P(\text{type } i \mid \text{occurs at } s)\, ds = \frac{1}{t}\int_0^t P_i(s)\, ds$ $\quad i = 1, \dots k.$

"Roll n dice w/ above prob's"

$\Rightarrow P(N_1(t)=n_1, \dots N_k(t)=n_k \mid N(t)=n) = $ multinomial

$$= \frac{n!}{n_1! n_2! \dots n_k!} \, p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$$

Plug into (*) to give:

$$P(N_1(t)=n_1, \dots N_k(t)=n_k) = \frac{\cancel{n!}}{n_1! \dots n_k!} p_1^{n_1} \cdots p_k^{n_k} \cdot \frac{(\lambda t)^n}{\cancel{n!}} \cdot e^{-\lambda t}$$

$$= \frac{(\lambda t p_1)^{n_1}}{n_1!} e^{-\lambda t p_1} \cdots \frac{(\lambda t)^{n_k}}{n_k!} e^{-\lambda t p_k}$$

i.e., $N_1(t) \dots N_k(t)$ are indep w/
$\mathbb{E}\, N_i(t) = \lambda t p_i = \lambda \int_0^t P_i(s)\, ds.$ ✓