Lecture 18

Note Title 2018-02-25

Poisson process continued.

Q: Robins & blackbirds arrive at a feeder according to indep Poisson processes $\{R(t): t \geq 0\}$ .. $\{B(t): t \geq 0\}$ w/ rates r & b.

Find

1. P(1st 2 birds are robins)
2. Dist. of total # of birds at time t
3. Dist. of blackbirds at time t given total # of birds is n.

A: 1. $P(\text{first bird is robin}) = P(Exp(r) < Exp(b))$

$$= \frac{r}{r+b}$$

By memorylessness,

$$P(\text{first 2 are robins}) = \left(\frac{r}{r+b}\right)^2$$

Similarly, $P(\underset{\text{robin}}{R}\,\underset{\text{black bird}}{B}\,RR) = \frac{r}{r+b} \cdot \frac{b}{r+b} \cdot \frac{r}{r+b} \cdot \frac{r}{r+b}$

2. $\{R(t)+B(t): t \geq 0\}$ is a poisson process w/ rate $r+b$. Thus,

$R(t)+B(t) \sim \text{Poisson}(t(r+b))$

3. $$P(B(t)=k \mid R(t)+B(t)=n) = \frac{P(B(t)=k,\ R(t)=n-k)}{P(R(t)+B(t)=n)}$$

$$= \frac{\frac{b^k}{k!} \cdot \cancel{e^{-b}} \cdot \frac{r^{n-k}}{(n-k)!} \cdot \cancel{e^{-r}}}{\frac{(r+b)^n}{n!} \cdot \cancel{e^{-(r+b)}}} = \binom{n}{k}\left(\frac{b}{r+b}\right)^k \cdot \left(\frac{r}{r+b}\right)^{n-k}$$

so $\text{Bin}\left(n, \frac{b}{r+b}\right)$

Q: Let $X \sim \text{Poisson}(\lambda)$ be # of games played between Alice and Bob. Suppose Alice wins each game independently w/ prob p.

Let A be # of games Alice wins
" B " " " " Bob ".

① What is dist of A?

② Are A & B indep?

① $$P(A=k) = \sum_{n \geq k} P(A=k \mid X=n) \cdot P(X=n)$$

$$= \sum_{n\geq k} \frac{\cancel{n!}}{k!(n-k)!} \cdot p^k (1-p)^{n-k} \cdot \frac{\lambda^n}{\cancel{n!}} \cdot e^{-\lambda}$$

$$= \frac{p^k}{k!} \lambda^k e^{-\lambda} \sum_{n \geq k} \frac{(1-p)^{n-k}}{(n-k)!} \lambda^{n-k}$$

$$= \frac{p^k}{k!} \lambda^k e^{-\lambda} \cdot \cancel{\sum_{n\geq 0} \frac{((1-p)\lambda)^n}{n!}} \quad e^{(1-p)\lambda}$$

$$= \frac{p^k}{k!} \lambda^k e^{-p\lambda} \quad \text{so} \quad \text{Poisson}(p\lambda)$$

② Indep! (Pf: Exercize for you)

These properties extend to the case of Pois. processes:

Prop Let $\{N(t) : t \geq 0\}$ be a rate λ Pois. proc. & suppose its events are independently

$$\begin{cases} \text{Type 1 w/ prob } p \\ \text{Type 2 w/ prob } 1-p. \end{cases}$$

Let $N_1(t)$ = # of Type-1 events by time t
$N_2(t)$ = # " Type-2 " " " "

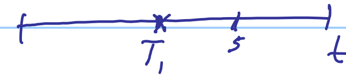
Then N_1 & N_2 are indep. Pois. processes w/ rates λp & $\lambda(1-p)$.

Pf See text.

Conditional dist. of arrival times

Q: Let bus arrival times be a rate λ Poisson process, N. Suppose 1 bus arrived in $[0, t]$. (I.e. condition on event $N(t)=1$.) What is the dist of the time it arrived?

A: Let T_1 = first arrival time.



$$P(T_1 < s \mid N(t)=1) = \frac{P(T_1 < s,\ N(t)=1)}{P(N(t)=1)} = \frac{P(N(s)=1,\ N(t)-N(s)=0)}{P(N(t)=1)}$$

$$= \frac{\lambda s \cdot e^{-\lambda s} \cdot e^{-\lambda(t-s)}}{\lambda t\, e^{-\lambda t}} = \boxed{\frac{s}{t}}$$

i.e. Uniform$[0, t]$