Lecture 17

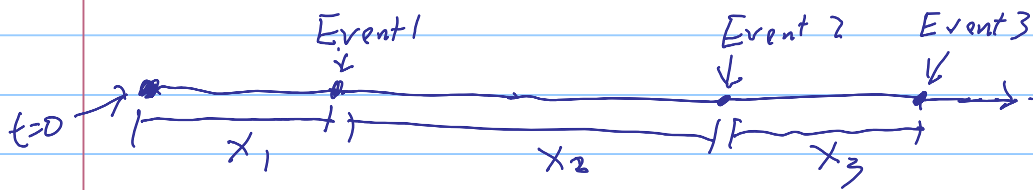
Note Title 2018-02-15

Poisson processes (formal definitions).

Def 1

Let $X_1, X_2, \ldots \overset{iid}{\sim} Exp(\lambda)$.

X_i = time it takes next event to occur.



Poisson process w/ rate λ:
$N(t)$ = # of events that have occured by time t.
i.e. $N(t) \geq k \iff \underbrace{X_1 + X_2 + \ldots X_k \leq t}_{k\text{th event occured by time } t}$

Second def of Poisson process needs:

$\boxed{\text{Def (little o notation) We say a fcn } f \text{ is } o(h) \text{ if } \lim_{h \to 0} \frac{f(h)}{h} = 0.}$

Ex) $f(h) = h^2$ so $\frac{h^2}{h} = h \to 0$

Ex) $e^h = 1 + h + o(h)$

why? Taylor series: $e^h = 1 + h + \underbrace{\frac{h^2}{2} + \frac{h^3}{3!} + \ldots}_{o(h)}$

Ex) $e^h \cdot (1+h) = (1 + h + o(h))(1+h) = 1 + 2h + o(h)$

Def 2 (Poisson process) A counting process $\{N(t); t \geq 0\}$ is a rate λ Poisson process if:

(i) $N(0) = 0$

(ii) Indep. increments i.e. $\underbrace{N(t) - N(s)}_{\# \text{ of events in } [s,t]}$ is indep of

$\underbrace{N(u)-N(v)}_{\substack{\#\text{ of}\\ \text{events in }[u,v]}}$ for $t \geq s \geq u \geq v$.

(iii) $P(N(t+h)-N(t)=1) = \lambda h + o(h)$

(iv) $P(N(t+h)-N(t)\geq 2) = o(h)$

<u>Prop</u>: ① Both definitions are equivalent.

② $N(t)-N(s) \sim \text{Poisson}(\lambda(t-s))$

<u>Pf</u>: See text.

Q: Let $X \sim \text{Poisson}(\alpha)$, $Y \sim \text{Poisson}(\beta)$ be indep.
Dist of $X+Y$?

A: Consider rate-1 poisson process $\{N(t): t \geq 0\}$
Let $X = N(\alpha) - \underbrace{\cancel{N(0)}}_{0} = N(\alpha) \sim \text{Poiss}(\alpha)$

$Y = N(\beta+\alpha) - N(\alpha) \sim \text{Poiss}(\beta+\alpha-\alpha) = \text{Poiss}(\beta)$

indep since increments are indep.

$X+Y = N(\alpha) + N(\beta+\alpha) - N(\alpha) = N(\beta+\alpha) \sim \text{Poiss}(\alpha+\beta)$.

<u>Prop</u> Let $\{N_1(t): t \geq 0\}$, $\{N_2(t): t \geq 0\}$ be indep pois proc. w/ rates λ_1 & λ_2.

Let $N(t) = N_1(t) + N_2(t)$. Then $\{N(t): t \geq 0\}$ is a rate $\lambda_1+\lambda_2$ Poiss. process.

<u>Pf</u>: We need to show that properties (i), (ii), (iii), (iv) of Def 2 hold.

(i): $N(0) = N_1(0) + N_2(0) = 0$

(ii) inherits indep increments

(iii) Write $\Delta N = N(t+h) - N(t)$

$$P(\Delta N=1) = P(\Delta N_1 + \Delta N_2 = 1) = P(\Delta N_1 = 0)\cdot P(\Delta N_2 = 1) + P(\Delta N_1 = 1)\cdot P(\Delta N_2 = 0)$$

$$= (1 - P(\Delta N_1 > 0))\cdot P(\Delta N_2 = 1) + (1 - P(\Delta N_2 > 0))\cdot P(\Delta N_1 = 1)$$

$$= (1 - \lambda_1\cdot h + o(h))(\lambda_2\cdot h + o(h)) + (1 - \lambda_2\cdot h + o(h))(\lambda_1\cdot h + o(h))$$

$$= \lambda_2 h + \lambda_1\cdot h + o(h)$$ ✓

(iv) $$P(\Delta N = 0) = P(\Delta N_1 = 0)\cdot P(\Delta N_2 = 0) = (1 - \lambda_1\cdot h + o(h))(1 - \lambda_2\cdot h \cdot o(h)) = 1 - (\lambda_1 + \lambda_2)h + o(h)$$

$$\Rightarrow P(\Delta N \geq 2) = 1 - P(\Delta N = 1) - P(\Delta N = 0) = 1 - [(\lambda_1 + \lambda_2)h + o(h)] - [1 - (\lambda_1 + \lambda_2)h + o(h)]$$

$$= o(h).$$ ✓