Lecture 16

Note Title 2018-02-08

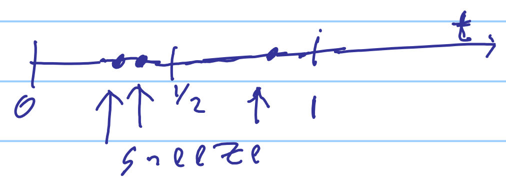
Poisson processes

Goal: random model for the following as a function of time

- cars passing on highway
- earthquakes
- sneezes
- ⋮

$N(t)$, $t \geq 0$ $(t \in \mathbb{R}_+)$

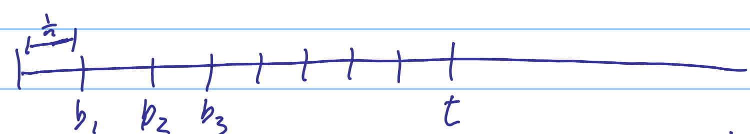
$N(t)$ = number of occurences in $[0, t]$.

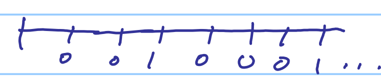


$N(1) = 3$

$N(\frac{1}{2}) = 2$

First attempt: discrete model. Discretize R_+





Let $b_i \overset{\text{indep}}{\sim} \text{Bernoulli}\left(\frac{\lambda}{n}\right)$ intensity parameter

Let $N(t) = \sum_{i=1}^{nt} b_i$

Q: ① What is dist of $N(t)$?

② What is dist of waiting time to first event?

$T = n \cdot \min\{i : b_i = 1\}$

A:

① $N(t) \sim Bin\left(\frac{\lambda}{n}, nt\right)$ (assuming $t \in \frac{1}{n}\mathbb{Z}_+$)

② $T = \frac{1}{n} Geom\left(\frac{\lambda}{n}\right)$

Q: What happens to these distributions as $n \to \infty$.

① $P(N(t)=k) = P\left(Bin\left(\frac{\lambda}{n}, nt\right)=k\right)$ for $t \in \mathbb{Z}_+$

$$= \binom{nt}{k} \cdot \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{nt-k}$$

$$= \frac{(nt)!}{k!(nt-k)!} \cdot \left(\frac{\lambda}{n}\right)^k \left(1-\frac{\lambda}{n}\right)^{nt-k}$$

$$= \frac{1}{k!} \frac{nt \cdot (nt-1)(nt-2)\cdots(nt-k+1)}{n^k} \cdot \lambda^k \cdot \left(1-\frac{\lambda}{n}\right)^{nt-k}$$

$$\Rightarrow \lim_{n\to\infty} P(N(t)=k) = \frac{1}{k!} \cdot t^k \lambda^k e^{-\lambda t} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Thus $N(t) \xrightarrow{dist} Poisson(\lambda t)$

Def We say $X \sim Poiss(\lambda)$ if

$$P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda} \qquad k=0,1,\ldots$$

② $P(T>t) = P\left(\frac{1}{n} Geom\left(\frac{\lambda}{n}\right) > t\right)$

$$= P\left(Geom\left(\frac{\lambda}{n}\right) \geq tn\right)$$

$$= \sum_{k=tn}^{\infty} \left(1-\frac{\lambda}{n}\right)^k \cdot \frac{\lambda}{n}$$ assuming $t \in \mathbb{Z}_+$

$$= \frac{\lambda}{n} \frac{\left(1-\frac{\lambda}{n}\right)^{tn}}{1-\left(1-\frac{\lambda}{n}\right)}$$

$$= \left(1-\frac{\lambda}{n}\right)^{tn}$$

Thus $\lim_{n\to\infty} P(T>t) = \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{tn} = e^{-\lambda t}$

$\Rightarrow$ $T \xrightarrow{dist} Exp(\lambda)$

Continuous case

We follow this intuition to define properties of a Poisson process.

Let $N(t)$, $t \geq 0$, be a Poisson process w/ intensity λ. Then

① For $t > s$, $N(t) - N(s) \sim \text{Poisson}(\lambda t) = \text{Pois}(\lambda t)$

② The waiting time between events T = time it takes for $N(t)$ to increase by 1 $\sim \text{Exp}(\lambda)$

③ For $t > s \geq u > v$

$N(t) - N(s)$ is indep of $N(u) - N(v)$ (independent increments)