Lecture 15

Note Title 2018-02-06

Before: Fixed jump times

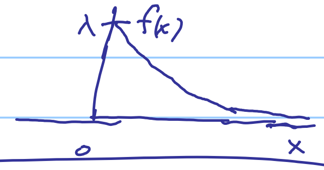
Now: Random jump times (tricky)

Next topic: <u>Poisson processes</u>. (Chapter 5)

Warm up: Review exponential r.v.'s.

Def We call X an exponential r.v. w/ parameter λ, i.e. $X \sim Exp(\lambda)$, if X has density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$



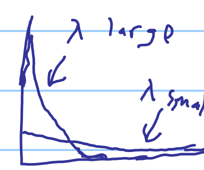
Rmk Typically models waiting time. Continuous analog of geometric r.v. Y. The latter has pmf $P(Y=n) = (1-p)^{n-1} \cdot p \approx e^{-np} \cdot p$ for n large, p small.

<u>Properties of $X \sim Exp(\lambda)$</u>

- $P(X > t) = \int_t^\infty \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_t^\infty = e^{-\lambda t}$.

- $\mathbb{E} X = \int_0^\infty P(X>t) = \int_0^\infty e^{-\lambda t} = -\frac{1}{\lambda} e^{-\lambda t}\Big|_0^\infty = \frac{1}{\lambda}$.

  true for any non-neg r.v



- $Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2 = \dots = \frac{1}{\lambda^2}$

  exercize for you

Q: Let $X \sim Exp(\frac{1}{10})$ be the waiting time for the bus. You've been waiting 10 minutes. On average, how much longer will you have to wait?

$$P(\underbrace{X > 10+t}_{A} \mid \underbrace{X > 10}_{B}) = \frac{P(A \cap B)}{P(B)} = \frac{P(X > 10+t)}{P(X > 10)} = \frac{e^{-\frac{10+t}{10}}}{e^{-\frac{10}{10}}}$$

$= e^{-\frac{t}{10}} = P(X > t)$. I.e. remaining waiting time has same $Exp(\frac{1}{10})$ dist.

Thus $\mathbb{E}[\overbrace{X - 10}^{\text{time remaining}} \mid X > 10] = 10$

"Bus doesn't care if you've been waiting 10 minutes."

More generally, exponential dist is memoryless, i.e.

if $X \sim Exp(\lambda)$, then $P(X > s+t \mid X > s) = P(X > t)$

Ex) You enter bank which has 2 tellers, currently servicing Yang & Lee. Service times are iid $Exp(\lambda)$. What is the chance you leave last?

A: One of the customers leaves first, say Yang. By memorylessness, Lee's remaining time is $Exp(\lambda)$. So is yours, even though you're just starting. Thus, by symmetry, you leave last with prob $\frac{1}{2}$.

Minimum of 2 Exp r.v.'s. (First bus to arrive)

$X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$ are indep.

Claim: $X = \min(X_1, X_2) \sim Exp(\lambda_1 + \lambda_2)$

Pf: $P(X > t) = P(X_1 > t \ \& \ X_2 > t)$

$= P(X_1 > t) \cdot P(X_2 > t)$ by indep

$= e^{-\lambda_1 t} \cdot e^{-\lambda_2 t}$

$= e^{-(\lambda_1+\lambda_2)t} = P(Exp(\lambda_1+\lambda_2) > t)$ ∎

Q: Which bus arrives first?

$X_1 \sim Exp(\lambda_1)$, $X_2 \sim Exp(\lambda_2)$

$$P(X_1 > X_2) = \mathbb{E}\, P(X_1 > X_2 \mid X_2) = \int_0^\infty P(X_1 > x_2 \mid X_2 = x) \cdot \lambda_2 e^{-\lambda_2 x} dx$$

$$= \int_0^\infty e^{-\lambda_1 x} \cdot \lambda_2 e^{-\lambda_2 x}\, dx = \lambda_2 \cdot \mathbb{E}\, Exp(\lambda_1 + \lambda_2)$$

$$= \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$