Lecture 14

Note Title 2018-02-04

Markov Chain Monte Carlo (MCMC)

- Monte Carlo (Explain bio-math project)

  Goal: Estimate $\mathbb{E}\, h(X)$

  function of interest high dimensional random vector.

Challenge
- h is hard to compute and/or
- X has many states so

$$\mathbb{E}\, h(X) = \sum_j h(x_j) \cdot P(X = x_j)$$ is computationally prohibitive.

Soln: "Simulate" several copies of $h(X)$

$$\mathbb{E}\, h(X) \approx \text{sample mean} = \frac{1}{n}\sum_{i=1}^{n} h(X_i)$$

law of large #'s indep copy of X.

Challenge: Sometimes producing a high-dim random vector is computationally infeasible.

Soln: MCMC Find an irreducible Markov chain w/ stationary dist π where $\pi_j = P(X = x_j)$. Take

$$\mathbb{E}\, h(X) \approx \frac{1}{n}\sum_{i=1}^{n} h(X_i) \qquad (*)$$

state of M.C. after i steps

$$\pi_j = \lim_{n\to\infty}\left(\begin{array}{l}\text{proportion of time}\\ \text{state spends in state } i\end{array}\right)$$

Thus, $$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} h(X_i) = \sum_j \pi_j \cdot h(x_j) = \sum_j P(X=x_j)\, h(x_j) = \mathbb{E}\, h(X)$$

i.e. RHS of $(*) \to \mathbb{E}\, h(X)$.

Key challenge: Given a stationary dist π, find a MC with that stationary dist.

Soln: Hastings-Metropolis algorithm.

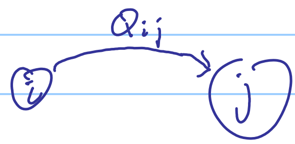
Input: π, Q ← trans. matrix for some irreducible MC. $(Y_n)_{n\geq 0}$ w/ commensurate state space.

$\uparrow$ non zero entries

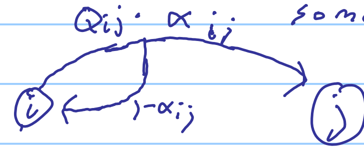
Output: Irreducible MC (X_n) w/ transition matrix P satisfying $\pi P = \pi$.

Works by adding "chute" to Y_n

Y_n



X_n



some extra resistance α_{ij}
Value TBD

$$P_{ij} = \begin{cases} Q_{ij} \cdot \alpha_{ij} \\ Q_{ii} + \sum_{j \neq i} Q_{ij}(1-\alpha_{ij}) \end{cases}$$

Desire: $\pi_i P_{ij} = \pi_j \cdot P_{ji}$

$$\pi_i \cdot Q_{ij} \alpha_{ij} = \pi_j \, Q_{ji} \alpha_{ji} \qquad (*)$$

Soln: $\alpha_{ij} = \min\left(\frac{\pi_j Q_{ji}}{\pi_i Q_{ji}}, 1 \right)$

Indeed, if $\pi_j Q_{ji} \leq \pi_i Q_{ji}$

then $\alpha_{ij} = \frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}$, $\alpha_{ji} = 1$ & $(*)$ is satisfied.

Similarly if $\pi_j Q_{ji} > \pi_i Q_{ji}$.