Lecture 13

Note Title 2018-02-01

Recall, $P(\text{ultimate extinction}) = \lim_{n\to\infty} P(Z_n=0) = \lim_{n\to\infty} G_n(0)$.

$$\boxed{\begin{array}{l}\textbf{Thm}\quad \alpha = P(\text{ultimate extinction}) = \begin{pmatrix}\text{smallest non-negative} \\ \text{root of } s=G(s)\end{pmatrix}. \\ \text{Also,} \\ \quad \text{let } \mu = \mathbb{E}\,\xi,\ \sigma^2 = Var(\xi). \\ \text{Then } \alpha = 1 \text{ if } \mu<1 \ \&\ \alpha<1 \text{ if } \mu>1. \\ \quad \text{If } \sigma^2>0 \ \&\ \mu=1, \text{ then } \alpha=1.\end{array}}$$

Not deterministic.

Ex) $\xi \sim Bin(2,p)$. $\qquad G(s) = (1-p+p\cdot s)^2$, $\mu = 2p$

$\sigma^2 = 2p(1-p)$

Solve $s = G(s) = (1-p)^2 + 2(1-p)p\cdot s + p^2 s^2$

$p^2 s^2 + [2p(1-p)-1]\cdot s + (1-p)^2 = 0$

Note: $G(1)=1$, $s=1$ is a root. Factor $s-1$ out:

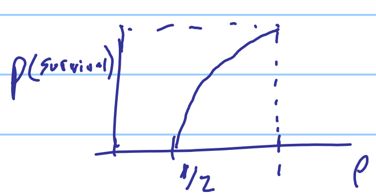
$$(s-1)(p^2 s - (1-p)^2) = 0$$

$\Rightarrow s = 1$ or $\dfrac{(1-p)^2}{p^2}$

We take smallest non-neg root:

$$\Rightarrow \alpha = \begin{cases} 1 & \text{if } p \le \frac{1}{2} \quad \text{i.e. } \mu \le 1 \\ \frac{(1-p)^2}{p^2} & \text{if } p > \frac{1}{2} \quad \text{i.e. } \mu > 1 \end{cases}$$

$$\Rightarrow P(\text{survival}) = \begin{cases} 0 & \text{if } p \le \frac{1}{2} \\ 1 - \frac{(1-p)^2}{p^2} & \text{if } p > \frac{1}{2} \end{cases}$$



Proof of thm:

Claim 1: α obeys $\alpha = G(\alpha)$.

Pf: Let $\alpha_n := P(Z_n = 0)$.

Recall $\alpha = \lim_{n\to\infty} \alpha_n$, $\alpha_n = G_n(0) = G(G_{n-1}(0)) = G(\alpha_{n-1})$

Take $\lim_{n\to\infty}$ on both sides $\Rightarrow \alpha = G(\alpha)$

Claim 2: α is smallest non-neg sol'n to $\alpha = G(\alpha)$.

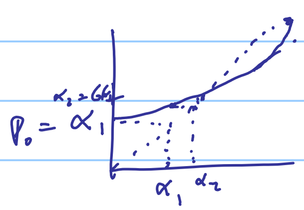
Pf: Let β be an arbitrary non-neg sol'n $\beta = G(\beta)$,
G is non-decreasing i.e. if $x \le y$, $G(x) \le G(y)$, so

$$\alpha_1 = G(0) \le G(\beta) = \beta \qquad \Rightarrow \alpha_1 \le \beta$$
$$\alpha_2 = G(\alpha_1) \le G(\beta) = \beta$$
$$\alpha_3 = G(\alpha_2) \le G(\beta) = \beta$$
$$\vdots$$
$$\alpha_n \le \beta \qquad \forall n$$
$$\Rightarrow \alpha = \lim_{n\to\infty} \alpha_n \le \beta. \quad \blacksquare$$

Claim 3: $\alpha = 1$ if $\mu \le 1$, $\alpha < 1$ if $\mu > 1$.

Pf:

$\alpha_n = G(\alpha_{n-1})$



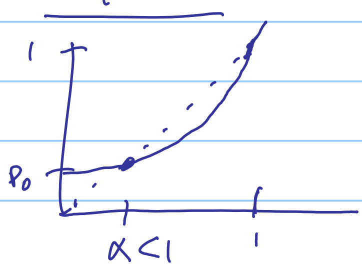
Does G hit $45°$ line?

Recall: $G''(s) = \mathbb{E}\, z(z-1) s^{z-2} \ge 0. \Rightarrow G$ is cvx.

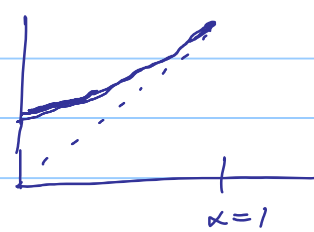
$\mu = G'(1)$,

$1 = G(1)$

$\mu < 1$



$\mu > 1$



Claim 4 $\mu = 1$, $\sigma > 0 \Rightarrow \alpha = 1$.

$G''(1) = \sigma^2 - \cancel{\mu} + \cancel{\mu^2} > 0$ $\quad$ ($G'(s) < 1$ for $s < 1$).