Note Title 2015-10-26

Majorizing measures theorem

Recall: Let X_t, $t \in T$ be a centered Gaussian process w/ assoc. metric $d(s,t) := \|X_s - X_t\|_{\psi_2}$. Then

$$c\,\gamma_2(T,d) \le \mathbb{E} \sup_{t \in T} X_t \le C \cdot \gamma_2(T,d)$$

Talagrand's Majorizing measures theorem

holds for sub Gaussian processes

Today we give some tools needed to prove the lower bound.

Recall:

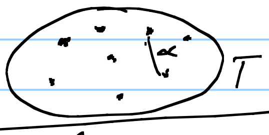
Def Let T be a set w/ metric d. We say a sequence of subsets $\{T_i\}_{i \ge 0}$ are admissible if $|T_0| = 1$, $|T_n| \le 2^{2^n}$, & $T_0 \subseteq T_1 \subseteq \ldots \subseteq T$. We define:

$$\gamma_2(T,d) := \inf \sup_{t \in T} \sum_{n \ge 0} 2^{n/2} d(t, T_n)$$

where the inf is taken over all admissible sequences of subsets.

Some Gaussian tools:

① Lower bound $\mathbb{E} \sup X_t$ via packing:



Theorem: (Sudakov ≠) Let X_t, $t \in T$ be a Gaussian process w/ $d(s,t) \ge \alpha$ for each distinct pair $s,t \in T$. Then

$$\mathbb{E} \sup_{t \in T} X_t \ge c\alpha \sqrt{\log |T|}$$

Proof: Consider another Gaussian process Y_t which consists of iid $cN(0, \alpha^2)$ elements.

Note: • $\mathbb{E}(Y_t - Y_s)^2 = \mathbb{E}\, 2c^2 N(0,\alpha^2) = 2c^2\alpha^2$

• $\mathbb{E}(X_t - X_s)^2 = C' \|X_t - X_s\|_{\psi_2}^2 = C'\alpha^2$

Choose c small enough and apply Slepian's $\neq$ to give

$$\mathbb{E} \sup_{t\in T} X_t \geq \mathbb{E} \sup_{t \in T} Y_t \geq c\sqrt{\log|T|}\;\alpha$$

QED

We also need the following version of Gaussian concentration:

$$\boxed{\textbf{Thm (Borell-TIS} \neq\textbf{)} \text{ Let } X_t,\ t\in T, \text{ be a Gaussian process. Then } \left\| \sup_{t\in T} X_t - \mathbb{E}\sup_{t\in T} X_t \right\|_{\psi_2} \leq c \sup_{t\in T} \mathbb{E} X_t^2}$$

"Deviation from expectation is controlled w/ maximum variance, independent of $|T|$."

Proof We assume $|T| = n < \infty$ for simplicity.

$$X = (X_t)_{t\in T} = (X_1, X_2, \ldots, X_n) \sim N(0, \Sigma)$$

for some Σ, Σ may be decomposed as $\Sigma = AA^T$ for some matrix A. Then

$$X \overset{dist}{=} Ag \quad \text{where } g \sim N(0, I_n)$$

$$\sup_{t\in T} X_t \overset{dist}{=} \max(Ag)$$

Note: the mapping $g \mapsto \max(Ag)$ has

Lipschitz constant $\|A\|_{2\to\infty} = \max_i \|a_i\|_2$

where a_i^T is the ith row of A.

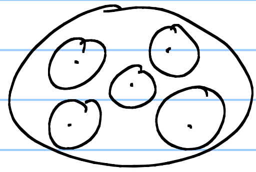
Also note $\mathbb{E}\langle a_i, g\rangle^2 = \|a_i\|_2^2$.

Thus, by Gaussian concentration,

$$\Big\| \underset{\sup_{t\in T} X_t}{\underset{\uparrow}{\max Ag}} - \underset{\mathbb{E}\sup_{t\in T} X_t}{\underset{\uparrow}{\mathbb{E}\max(Ag)}} \Big\|_{\psi_2} \le C \underset{\sqrt{\sup \mathbb{E}X_t^2}}{\underset{\uparrow}{\sqrt{\max \mathbb{E}\langle a_i, g\rangle^2}}}$$

QED

We can use the Borel-TIS $\neq$ to robustify sudakov $\neq$.



Notation: Given a Gaussian process X_t, $t\in T$, and a subset $A\subseteq T$, let $g(A) := \mathbb{E}\sup_{t\in A} X_t$.

Let $B(t,R) := \{s\in T : d(s,t)\le R\}$.

$\boxed{\text{\underline{Corollary} (Robust sudakov } \neq\text{) Let } X_t,\ t\in T, \text{ be a Gaussian process and } t_1,\dots,t_m\in T \text{ be s.t. } d(t_i,t_j)\ge\alpha \text{ for } i\neq j. \text{ Then } g(T) \ge c\alpha\sqrt{\log m} + \min_{i=1,\dots,m} g(B(t_i, c\alpha))}$

$\boxed{\text{\underline{Proof} Exercize}^{+}}$

We are now in position to prove the lower bound:

$$\mathbb{E} \sup_{t \in T} X_t \geq c\, \gamma_2(T, d)$$

Claim: Let $F : 2^T \to \mathbb{R}$ be any functional satisfying

① $F(A) \leq F(B)$ for $A \subseteq B \subseteq T$

② Let $t_1, \ldots, t_m \subseteq A \subseteq T$ satisfy $d(t_i, t_j) \geq \alpha$. Then

$$F(A) \geq c\, \alpha \sqrt{\log m} + \min_{i=1,\ldots,m} F(B(t_i, c\alpha))$$

Then $F(T) \geq \gamma_2(T, d)$.

If claim is true, then lower bound is proven by taking $F : A \mapsto \mathbb{E} \sup_{t \in A} X_t$.

$\boxed{\text{Extra stuff}}$ Prove claim

Hints:

① This is hard.

② For more hints, talk to me.